Stationarity I

Definition (Strict Stationarity)

The distribution of the random process has certain attributes that are the same everywhere. **Strict stationarity** indicates that for any number $k$ of any sites $x_1, x_2, \ldots, x_k$, the joint cumulative distribution of $(s(x_1), \ldots, s(x_k))$ remains the same under an arbitrary translation $h$:

$$P(s(x_1), \ldots, s(x_k)) = P(s(x_1 + h), \ldots, s(x_k + h))$$

If the random process is strictly stationary, its moments if they exist are also invariant under translations.

Stationarity II

Definition (Weak stationarity)

A process $s(x)$ is said **second order stationary** (or weakly stationary, or wide-sense stationary) when:

1. the mean of the process does not depend on $x$: $\mathbb{E}[s(x)] = \mu$
2. the variance of the process does not depend on $x$: $\mathbb{E}[(s(x) - \mu)^2] = \sigma^2$
3. the covariance between $s(x)$ and $s(x + h)$ only depends on $h$:

$$\text{Cov} [s(x), s(x + h)] = \mathbb{E} [(s(x) - \mathbb{E}[s(x)]) (s(x + h) - \mathbb{E}[s(x + h)])]$$
$$= \mathbb{E} [(s(x) - \mu) (s(x + h) - \mu)]$$
$$= C(h)$$

Note that $C(0) = \sigma^2$.

Stationarity III

- For a weakly stationary process, the autocorrelation can also be defined as:

$$\rho(h) = \frac{C(h)}{C(0)}$$

with $C(0)$ is the covariance at lag 0, i.e. $\sigma^2$.

- If a random field with the function $C(h)$ only dependent on the distance $||h||$ and not on its orientation, it is said to be isotropic.

Stationarity IV

Definition (Intrinsic Stationarity)

$s(x)$ is said to be an **intrinsic random function** such that:

$$\mathbb{E}[s(x + h) - s(x)] = 0$$

and

$$\text{Var}[s(x + h) - s(x)] = 2 \gamma(h)$$

The function $\gamma(h)$ is called the **variogram**.
Stationarity V

**Exercises:** For second-order stationary processes,

- Express $\gamma(h)$ w.r.t. $C(h)$.
- Express $\gamma(h)$ w.r.t. $\rho(h)$.
- Show that a process that is weakly stationary is also intrinsically stationary.

Brownian Motion & Ornstein-Uhlenbeck Processes

- A second order stationary process is also an intrinsic stationary process.
- But an intrinsic stationary process is not always a second order stationary process or a strictly stationary process.
- To illustrate the weakly stationary and intrinsic stationary, we look at the following processes:
  - Brownian Motion,
  - Ornstein-Uhlenbeck Process.

Brownian Motion I

**Definition (diffusion & Brownian Motion)**

A **diffusion** is a continuous time Stochastic Process $s(t)$ with the following properties:

- $s(0) = 0$,
- $s(t)$ has independent increments,
- $P(s(t_2) | s(t_1))$ has a density function $f(s(t_2)|t_1, t_2, s(t_1))$.

**Standard Brownian motion** is a diffusion $s(t)$, $t \geq 0$ satisfying the following:

- $s(0) = 0$,
- $s(t)$ has independent increments.
- For $t_2 > t_1$, $s(t_2) - s(t_1) \sim \mathcal{N}(0, \sigma^2(t_2 - t_1))$.

Brownian Motion II

- $s(t)$ has independent increments means that for all times $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the increments $s(t_n) - s(t_{n-1}), s(t_{n-1}) - s(t_{n-2}), \cdots, s(t_2) - s(t_1)$ are independent random variables.

- For $t_2 > t_1$, $s(t_2) - s(t_1) \sim \mathcal{N}(0, \sigma^2(t_2 - t_1))$ is equivalent to $s(t_2) | s(t_1) \sim \mathcal{N}(s(t_1), \sigma^2(t_2 - t_1))$.

$s(t_2)$ given $s(t_1)$ is normally distributed with mean $s(t_1)$ and variance $\sigma^2(t_2 - t_1)$. 
Brownian Motion III

**Exercises:**
- Show that a standard brownian motion is intrinsically stationary
- Show that a standard brownian motion is not weakly stationary

Ornstein-Uhlenbeck Process I

**Definition (Ornstein-Uhlenbeck)**

Let $s(t)$ be standard Brownian motion. The process

$$V(t) = e^{-t}s(e^{2t})$$

is called the **Ornstein-Uhlenbeck** process.

Ornstein-Uhlenbeck Process II

Show that $V(t)$ is intrinsically stationary and weakly stationary.

Brownian Motion and Ornstein-Uhlenbeck processes

**Historical remarks:**
- Robert Brown observes ceaseless irregular motion of small particles in a fluid. Motion explained by believing particles to be alive (1827-1829).
- Goul puts forward a kinetic theory to explain the motion; it is due to rapid bombardment of a huge number of fluid molecules (1860).
- Einstein presents the theory of “Brownian motion” (c. 1900). At the same time (c. 1900), Bachelier defines it to model stock options.
- The Ornstein-Uhlenbeck process was proposed by Uhlenbeck and Ornstein (1930) in a physical modelling context, as an alternative to Brownian Motion. The model has been used since in a wide variety of applications areas e.g. in finance (see Vasicek (1977) interest rate model).