Consider the following dataset

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t^{(i)} )</td>
<td>2</td>
<td>2.5</td>
<td>6</td>
<td>4</td>
<td>4.5</td>
<td>2.5</td>
<td>2.4</td>
<td>4.4</td>
<td>8</td>
<td>7.5</td>
</tr>
<tr>
<td>( s^{(i)} )</td>
<td>1</td>
<td>1.5</td>
<td>1</td>
<td>5.4</td>
<td>5.5</td>
<td>1.3</td>
<td>1.4</td>
<td>5.3</td>
<td>1</td>
<td>2.2</td>
</tr>
</tbody>
</table>

Table 1: Process \( s \) observed at times \( t \) (10 observations).

1. Compute the histograms \( H(s) \) and \( H(t) \) with
   (a) bin size \( b = 1 \) and intervals \([a; a + b]\) starting with \( a = 0 \).
   (b) bin size \( b = 2 \) and intervals \([a; a + b]\) starting with \( a = 0 \).
2. Compute the joint histogram \( H(t, s) \) with bin size \( b = 1 \) and intervals \([a; a + b]\) starting with \( a = 0 \) (both directions).
3. Compute the p.d.e. \( \hat{p}_b(s) \) and \( \hat{p}_b(t) \) from histograms \( H(s) \) and \( H(t) \) (with bin size \( b = 1 \)).
4. Write down the empirical density estimate \( \hat{p}_e(s) \).
5. Write down the Kernel density estimates \( \hat{p}(s) \) with the Gaussian kernel\(^1\) and bandwidth \( h = 1 \).
6. Check that \( \hat{p}(s) \) is the result of a convolution between the kernel function and \( \hat{p}_e \).
7. Compute the kernel density estimates \( \hat{p}(s, t) \) and \( \hat{p}(t) \) (with Gaussian kernel, \( h = 1 \)).
8. Compute the Nadaraya Watson estimators using the Kernel density estimates \( \hat{p}(s, t) \) and \( \hat{p}(t) \).
9. Predict \( s \) at \( t = 7 \) and \( t = 3 \) with NW estimator.

\(^1\)alternative kernels can be used [https://en.wikipedia.org/wiki/Kernel_(statistics)#Kernel_functions_in_common_use](https://en.wikipedia.org/wiki/Kernel_(statistics)#Kernel_functions_in_common_use)
1. Solutions:

<table>
<thead>
<tr>
<th>intervals</th>
<th>[0,1)</th>
<th>[1,2)</th>
<th>[2,3)</th>
<th>[3,4)</th>
<th>[4,5)</th>
<th>[5,6)</th>
<th>[6,7)</th>
<th>[7,8)</th>
<th>[8,9)</th>
<th>[9,10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(t)$</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$H(s)$</td>
<td>0</td>
<td>6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: For $b = 1$

<table>
<thead>
<tr>
<th>intervals</th>
<th>[0,2)</th>
<th>[2,4)</th>
<th>[4,6)</th>
<th>[6,8)</th>
<th>[8,10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(t)$</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$H(s)$</td>
<td>6</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3: For $b = 2$

2. cf. Fig. 1

3. $\hat{p}_b(s) = \frac{H(s)}{10 \times b}$ and $\hat{p}_b(t) = \frac{H(t)}{10 \times b}$

4. 

\[ \hat{p}_e(s) = \frac{1}{10} \sum_{i=1}^{10} \delta(s - s^{(i)}) \]

5. 

\[ \hat{p}_e(s) = \frac{1}{10} \sum_{i=1}^{10} \mathcal{N}(s; s^{(i)}, 1) \]

6. 

\[ \hat{p}(s) = \mathcal{N}(0, h^2) * \hat{p}_e = \int \left( \frac{1}{\sqrt{2\pi}h} \exp \left( -\frac{(s - \tau)^2}{2h^2} \right) \right) \left( \frac{1}{10} \sum_{i=1}^{10} \delta(\tau - s^{(i)}) \right) \, d\tau \]

7. 

\[ \hat{p}(s, t) = \frac{1}{10} \sum_{i=1}^{10} \mathcal{N}(s; s^{(i)}, 1) \cdot \mathcal{N}(t; t^{(i)}, 1) \]

8. 

\[ E[s|t] = \frac{\sum_{i=1}^{10} s^{(i)} \cdot \mathcal{N}(t; t^{(i)}, 1)}{\sum_{i=1}^{10} \mathcal{N}(t; t^{(i)}, 1)} \]

9. 

\[ E[s|7] = \frac{\sum_{i=1}^{10} s^{(i)} \cdot \mathcal{N}(7; t^{(i)}, 1)}{\sum_{i=1}^{10} \mathcal{N}(7; t^{(i)}, 1)} = 1.6646 \]

and

\[ E[s|3] = \frac{\sum_{i=1}^{10} s^{(i)} \cdot \mathcal{N}(3; t^{(i)}, 1)}{\sum_{i=1}^{10} \mathcal{N}(3; t^{(i)}, 1)} = 2.4993 \]
Figure 1: Solution histograms.