Finite Difference Method

Example

Let consider the O.D.E. on the interval \( t \in \Delta = [0; 1] \)
\[ f(t, y, y') = y'(t) + y(t) = 0 \]
with the initial condition \( y(0) = 1 \).

We can solve this O.D.E.:
- **analytically** giving the exact solution:
  \[ y(t) = \exp(-t) \]
- **numerically** giving an approximate solution using:
  - Finite Difference Method
  - Finite Element Method

Finite Element Method I

Definition (Collocation method)

The **collocation method** imposes that the residual vanishes at \( n \) points i.e. \( R(t_i; a) = 0 \) for \( t_i \in \Delta \) and \( i = 1, \ldots, n \).

Example

We choose the basis \( \phi_i(t) = t^i \) and \( n = 2 \) so:
\[ \hat{y}(t) = 1 + a_1 \phi_1(t) + a_2 \phi_2(t) = 1 + a_1 \ t + a_2 \ t^2 \]
and
\[ R(t; a) = 1 + (1 + t) \ a_1 + (2t + t^2) \ a_2 \]

Finite Element Method II

Definition (Finite Element Method)

Consider an equation (O.D.E.) \( f(t, y, y') = 0 \) of a function \( y(t) \) defined on the interval \( t \in \Delta \):

- Define a trial solution of the form:
  \[ \hat{y}(t) = \phi_0 + \sum_{i=1}^{n} a_i \phi_i(t) \]
  where the functions \( \phi_i(t) \) are basis functions, and \( \phi_0 \) refers to a constant that usually can be inferred from initial conditions.
- Replace \( y(t) \) by \( \hat{y}(t) \) in the O.D.E. \( f(t, \hat{y}, \hat{y}') = R(t, a) \) (\( R \) is called the residual) and solve the coefficients \( \{a_i\}_{i=1}^{n} \).

The estimated \( \hat{y}(t) \) gives a numerical solution to the O.D.E. defined on the interval \( \Delta \).

Solving \( y' + y = 0 \) using collocation method I

Example

The collocation method imposes that the residual vanishes at \( n \) points i.e. \( R(t_i; a) = 0 \) for \( t_i \in \Delta \) and \( i = 1, \ldots, n \).

Example

We take \( t_1 = \frac{1}{2} \) and \( t_2 = \frac{3}{2} \) such that we impose \( R(t_1; a) = R(t_2; a) = 0 \). This gives us 2 equations:
\[ \begin{align*}
1 + (1 + \frac{1}{2}) \ a_1 + \left( \frac{3}{2} + \left( \frac{1}{2} \right)^2 \right) \ a_2 &= 0 \\
1 + (1 + \frac{3}{2}) \ a_1 + \left( \frac{3}{2} + \left( \frac{3}{2} \right)^2 \right) \ a_2 &= 0
\end{align*} \]

Solving in \( a_1 \) and \( a_2 \) gives:
\[ \hat{y}(t) = 1 - \frac{27}{29} t + \frac{9}{29} t^2 \]

Compare that with the true function \( y(t) = \exp(-t) \).

Solving \( y' + y = 0 \) using collocation method II

Example

We take \( t_1 = \frac{1}{2} \) and \( t_2 = \frac{3}{2} \) such that we impose \( R(t_1; a) = R(t_2; a) = 0 \). This gives us 2 equations:
\[ \begin{align*}
1 + (1 + \frac{1}{2}) \ a_1 + \left( \frac{3}{2} + \left( \frac{1}{2} \right)^2 \right) \ a_2 &= 0 \\
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Solving in \( a_1 \) and \( a_2 \) gives:
\[ \hat{y}(t) = 1 - \frac{27}{29} t + \frac{9}{29} t^2 \]

Compare that with the true function \( y(t) = \exp(-t) \).

Sub-domain method I

Definition (subdomain method)

Divide the interval \( \Delta \) of definition of the O.D.E. into subdomains \( \{\Delta_i\}_{i=1}^{n} \), and solve \( a \) such that:
\[ \frac{1}{\Delta_i} \int_{\Delta_i} R(t; a) \ dt = 0 \quad \forall i = 0, \ldots, n \]

Sub-domain method II

Example

We choose \( \Delta_1 = [0; \frac{1}{2}] \) and \( \Delta_2 = [\frac{1}{2}; 1] \) and:
\[ \frac{1}{\Delta_1} \int_{\Delta_1} R(t; a) \ dt = 2 \left[ \frac{1}{2} + \frac{5}{8} a_1 + \frac{7}{24} a_2 \right] = 0 \]
and
\[ \frac{1}{\Delta_2} \int_{\Delta_2} R(t; a) \ dt = 2 \left[ \frac{1}{2} + \frac{7}{8} a_1 + \frac{25}{24} a_2 \right] = 0 \]
Solving this system gives the solution:
\[ \hat{y}(t) = 1 - \frac{18}{19} t + \frac{6}{19} t^2 \]

Compare \( \hat{y}(t) \) estimated using subdomain method, with the exact solution \( y(t) \).
### Least-Squares Method I

**Definition (Least-Squares Method)**

We can use least square estimation to find the optimum values of the parameters $a$:

$$\hat{a} = \arg \min \left\{ \int_\Delta (R(t; a))^2 \, dt \right\}$$

or solve

$$\frac{\partial}{\partial a_i} \int_\Delta (R(t; a))^2 \, dt = 0 \quad \forall i$$

or

$$\int_\Delta \frac{\partial}{\partial a_i} R(t; a) \, dt = 0 \quad \forall i$$

### Galerkin Method I

**Definition (Galerkin Method)**

The Galerkin method imposes the following criterium to estimate $a$:

$$\int_\Delta R(t; a) \phi_i(t) \, dt = 0 \quad \forall i = 1, \ldots, n$$

**Example**

Find that the estimate of $\hat{y}(t)$ using Galerkin method is:

$$\hat{y}(t) = 1 - \frac{32}{35} t + \frac{2}{7} t^2$$

Compare that solution with the exact one $y(t)$.

### Exercises

Solve on the interval $[0; 1]$ the O.D.E.:

$$y''(t) + y(t) = 1$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$:

- analytically.
- numerically using
  - the collocation method
  - the subdomain method
  - the Galerkin method
- Draw all the solutions you found on a graph (use matlab).

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### Least-Squares Method II

**Example**

Exercise: Try to find $\hat{y}$ with the Least squares method.

### Galerkin Method II

Weighting function $W_i$ for:

- the Collocation method:
  $$W_i(t) = \delta(t - t_i)$$

- the Sub-domain method:
  $$W_i(t) = \begin{cases} 1 & \text{if } t \in \Delta_i \\ 0 & \text{otherwise} \end{cases}$$

- the Least Squares method:
  $$W_i(t) = 2 \frac{\partial R(t; a)}{\partial a_i}$$

- the Galerkin Method:
  $$W_i(t) = \frac{\partial \hat{y}}{\partial a_i} = \phi_i(t)$$
Remarks on FDM and FEM

**Finite Difference Method:** FDM approximates an operator (e.g., the derivative) and solves a problem on a set of points (the grid).

**Finite Element Method:** FEM uses exact operators but approximates the solution basis functions. Also, FE solves a problem on the interiors of grid cells (and optionally on the gridpoints as well).

Which basis functions? I

- Amongst the methods of weighted residuals, the Galerkin method is the most commonly used.
- MWR apply equally well to any family of basis functions \{\phi_i\}. However to be successful, the basis functions need to be chosen with care:
  - Global basis functions perform badly in fitting local behaviour, e.g.:
    \[ \phi_i(t) = t^i \]

Which basis functions? II

- More local basis functions can be defined so that there are non-zero only on small regions, e.g. piecewise linear (or hat) functions:

\[
\phi_i(t) = \begin{cases} 
\frac{t - t_i}{t_i - t_{i-1}} & t_{i-1} \leq t < t_i \\
\frac{t - t_i}{t_{i+1} - t_i} & t_i \leq t < t_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

Which basis functions? III