Axiom quantification Distributivity

Axiom quantification Distributivity: Provided each quantification is defined,

\[(\forall x)(R \land P) \equiv (\forall x)(R) \land (\forall x)(P)\]

Example:
It is clear that the following quantifications are well defined. Applying Axiom Distributivity gives:

\[\begin{align*}
(\forall x)(i^2 < 9 : i^2) + (\forall x)(i^2 < 9 : i^2) &= (\forall x)(i^2 < 9 : i^2 + i^2)
\end{align*}\]

Exercise: Application of Axioms

Solutions are in red.

1. Using Axiom Empty range, simplify the expression \([\forall x]([\forall y]y \notin (p \land (q \lor x))\).

   From the axiom Empty range, we know the solution of \((\forall x)F\) is the identity element \(x\) (or neutral element) of the operator \(\lor\), defined such that \(x \lor x = x\). For the operator \(\land\), the neutral element is \(\bot\) because \(p \land F = p\). Consequently \([\forall x]([\forall y]y \notin (p \land (q \lor x))\) = \(\bot\).

2. Using Axiom One-point rule, simplify the expression \([\forall x][i = 6 : (i + 1)]\).

   \([\forall x][i = 6 : (i + 1)] = 6 \cdot (6 + 1) = 42\)

3. Using Axiom quantification distributivity, simplify the expression

   \[\begin{align*}
   (\forall x)(2i < 10 : -i^2) + (\forall x)(2i < 10 : i^2)
   \end{align*}\]

   \[\begin{align*}
   (\forall x)(2i < 10 : -i^2) + (\forall x)(2i < 10 : i^2) &= (\forall x)(2i < 10 : -i^2 + i^2) = 0
   \end{align*}\]

Axiom Range split

Axiom Range split: Provided \(R \land S = F\) and each quantification is defined,

\[(\forall x)(R \lor S : P) = (\forall x)(R : P) \lor (\forall x)(S : P)\]

Example: 
Recalling even \(i\) is true if \(i\) is even and false otherwise,

\[\begin{align*}
(\forall x)[1 \leq i \leq 10 \land \text{even}(i)] + (\forall x)[1 \leq i \leq 10 \land \text{odd}(i)]
\end{align*}\]

General Axiom Range split

The axiom Range split has been defined with the condition \(R \land S = F\). This condition is relaxed in the following Axiom.

General Axiom Range split: Provided each quantification is defined,

\[(\forall x)(R \lor S : P) = (\forall x)(R : P) \lor (\forall x)(S : P)\]

Example: 
Let \(R\) be the statement \(1 \leq i \leq 10\) and \(S\) be the statement \(5 \leq i \leq 15\). The \(R \lor S\) is the statement \(1 \leq i \leq 15\) and \(R \land S\) is the statement \(5 \leq i \leq 10\). From Axiom Range split:

\[\begin{align*}
(\forall x)[1 \leq i \leq 15 : i^2] + (\forall x)[5 \leq i \leq 10 : i^2]
\end{align*}\]

\[\begin{align*}
(\forall x)[1 \leq i \leq 10 : i^2] + (\forall x)[5 \leq i \leq 15 : i^2]
\end{align*}\]

Axiom Range split for idempotent +

Definition:
An operator \(+\) is called idempotent if \(s \times s = s\) for all \(s\).

- The operators \(\land\) and \(\lor\) are idempotent.
- The operators \(+\) and \(-\) are not idempotent.

Axiom Range split for idempotent +: Provided each quantification is defined,

\[(\forall x)(R \lor S : P) = (\forall x)(R : P) \lor (\forall x)(S : P)\]

Exercise: Axioms Range split

1. Using (General) Axiom Range Split, rewrite the following

   \(\begin{align*}
   (i) \quad (\forall x)[1 \leq i \leq 10 \land 20 < i \leq 30 : i^2]
   (ii) \quad (\forall x)[1 \leq i \leq 5 \land 9 \land 7 \land 5 \leq i \leq 30 : i]
   (iii) \quad (\forall x)[1 \leq i \leq 10 \land 20 < i \leq 30 : i + 3)
   \end{align*}\)

2. Using (General) Axiom Range Split, rewrite the following

   \(\begin{align*}
   (i) \quad (\forall x)[1 \leq i \leq 12 : 2i] + (\forall x)[3 \leq i \leq 10 : 2i]
   (ii) \quad (\forall x)[1 \leq i \leq 20 : 4i + 3] + (\forall x)[10 \leq i \leq 11 : 4i + 3]
   (iii) \quad (\forall x)[1 \leq i \leq 9 : i] \times (\forall x)[2 \leq i \leq 17 : i]
   \end{align*}\)

   We cannot apply (General) Axiom Range Split to the following expression. Why?

   \(\begin{align*}
   (\forall x)[20 \leq i \leq 40 : i] \times (\forall x)[5 \leq i \leq 14 : 2i]
   \end{align*}\)
Solution: Axioms Range split

(i) Let \( R \) be \((1 \leq i \leq 10)\), and \( S \) be \((20 \leq i \leq 30)\), then \( R \land S = \emptyset \) and applying Axiom Range Split:
\[
(i+1 \leq i \leq 10 \lor 20 \leq i \leq 30) \land \neg \equiv (i+1 \leq i \leq 10) \lor (i+1 \leq i \leq 30) \land \neg
\]

(ii) Remember that \( \neg, <, >, \leq, \geq \) are conjunctural. Let \( Q \) be \((1 \leq i \leq 5)\), and \( S \) be \((i^2 - 36 \leq 30)\). \( S \) is equivalent to \((i^2 - 36) \lor (36 \leq 30)\), so \( S = F \). Consequently \( R \lor S \equiv R \lor F \equiv R \) and:
\[
(i+1 \leq i \leq 3 \lor i^2 - 36 \leq 30) \land \equiv (i+1 \leq i \leq 3) \land \equiv
\]

Exercise: Axiom Interchange of Dummies

1. Rewrite the following quantifications using Axiom Interchange of Dummies.

   (i) \([+i | 1 \leq i \leq 5 : (+j | 4 \leq j : 2 + j - 1)]\)

   (ii) \([+i | 1 \leq i \leq 10 : (+j | \text{even}: j)]\)

   (iii) \([\times i | \text{odd}: i : (\times j | \text{even}: j)]\)

Axiom Interchange of Dummies

Axiom Interchange of dummies: Provided each quantification is defined, that \( y \) does not occur in \( R \) and \( x \) does not occur in \( Q \):
\[
[\times x | \{ R \land y : (Q : P) \}] = [\times y | \{ R : (Q : P) \}]
\]

Example:
Since \( i \) does not occur in the expression \([1 \leq j \leq 5]\) and \( j \) does not occur in the expression \([1 \leq i \leq 10]\), we may use the above axiom:
\[
(+i | 1 \leq i \leq 10 : ( +j | 1 \leq j \leq 5 : 2 + j ))
\]

Exercise: Axiom Nesting

1. Rewrite the following quantifications using Axiom Nesting.

   (i) \([+i | 10 < i : i < j : i - j)]\)

   (ii) \([+i | 0 < i \land \text{even}(j) : i + j)]\)

   (iii) \([\times i | \text{odd}: i : (\times j | \text{even}: j)]\)

Axiom Nesting

Axiom Nesting: Provided that \( y \) does not occur in \( R \)
\[
[\times x | \{ R \land y : (Q : P) \}] = [\times y | \{ R : (Q : P) \}]
\]

Example:
Since \( j \) does not occur in the expression \([1 \leq i \leq 10]\) we may apply the above axiom.
\[
(+i | 1 \leq i \leq 10 \land 1 \leq i + j \leq 19 : 3i + 2j )
\]

Axiom Dummy renaming

Axiom Dummy renaming: Provided \( y \) does not occur in \( R \) and \( P \):
\[
[\times x | \{ R : (P : x) \}] = [\times y | \{ R[x := y] : P[x := y] \}]
\]

Example:
\[
(+i | 2 \leq i \leq 10 : i^2 ) = (+i | 2 \leq i \leq 10 : i^2)
\]
Theorem Change of dummy

Theorem Change of dummy: Provided \( y \) does not occur in \( R \) and \( P \), and \( f \) has an inverse,

\[ \forall x (R \rightarrow P) = \forall y (R[x := f(y)] \rightarrow P[x := f(y)]) \]

Example:
Let \( f \) be the function given by \( f(k) = k + 2 \). The inverse of \( f \) is \( f^{-1}(k) = k - 2 \). Applying this to the quantification from our last example,

\[ (+i/2 \leq i \leq 10 : i^2) = (+k/2 \leq (k+2) \leq 10 : (k+2)^2) \]

Exercise: Theorem Change of dummy

Apply Theorem Change of dummy to each of the following quantifications using the function beside each one.

1. \( f(x) = x + 4; (+i/4 \leq i \leq 10 : (i-4)^2) \)
2. \( f(x) = x + 5; (\times i/6 \leq i \leq i + 5) \)
3. \( f(x) = x^2; (+i:1 \leq i^2 \leq 81 : i) \)