Higher Dimensions

Multi Objective Decision Making
Practical Difficulties

• It is possible, in principle, to elicit the utility of anything.

• However, in practice it may be difficult to collapse a problem down to a single dimension.

• How much is your health worth?
Other methods

• Other methods exist to deal with multiple objectives or criteria. Among them are:
  – Trade off curves
  – Analytic Hierarchy Process (AHP)
  – Goal Programming

Objective

- An objective is a dimension upon which a decision maker places weight or importance.
- For example, it could be profit, or political acceptability or pollution.
- In the case of MODM, it is assumed that choices can at least be ordered for each objective.
Trade Off Curves

- It is assumed that the value of each course of action may be summarised as a real number for each objective.
- Each course of action can then be described by a point in multidimensional space.
- For simplicity, consider the case of a decision in two dimensions. For example ‘political acceptability’ and ‘profit.’
Dominated solutions

- A solution (or decision) is dominated if there exists a feasible solution better in at least one dimension, and at least as good in every other dimension.
- That is, $a$ is dominated if there exists $b$ s.t. $u_i(b) \geq u_i(a)$ all $i$ and $u_j(b) > u_j(a)$ some $j$. 
Pareto Optimality

- A solution, \( \mathbf{a} \), to a multi objective problem (mop), is said to be **Pareto optimal** if no other feasible solution is at least as good as \( \mathbf{a} \) wrt to every objective and strictly better than \( \mathbf{a} \) wrt at least one objective.
- That is \( \mathbf{a} \) is Pareto optimal if for all \( \mathbf{b} \) s.t. \( u_i(\mathbf{b}) > u_i(\mathbf{a}) \) then there exists \( j \) s.t. \( u_j(\mathbf{a}) > u_j(\mathbf{b}) \).
- Pareto optimal = not dominated.
ToC Defined

• The collection of points in the decision space that are Pareto optimal is termed the trade off curve.
• Once this has been identified, the final decision is to choose a point on the curve.
• The process of identifying the trade off curve can often help the decision process.
Example (see lab)

• Manufacturing problem, with 8 products having different resource use, profits and pollutions.

• Objective: Max Profit and Min Pollution.

• Method:
  – Unconstrain Pollution. Max profit.
  – Unconstrain Profit. Min Pollution.
  – Condition on Pollution Vals, Max Profit.
Analytic Hierarchy Process

- AHP is a structured way of ranking various options open to a decision maker.
- It requires a step by step process of pairwise comparisons of options against each of the objectives.
- Then the objectives themselves must be compared and weighted.
- Each objective can have a collection of sub-objectives on which this process can be carried out, from where the ‘hierarchy’ originates.
An example

• Consider a graduate with 3 job offers.
• Let us say that she has 4 criteria or objectives which she wishes to base her decision upon;
  – Salary
  – Attractiveness of City of job
  – Interesting Work
  – Closeness to Family
Method

• In a structured way, find weights for each of the 4 objectives (that sum to 1). Call these $w_1, \ldots, w_4$.

• Similarly, divide up a unit across each of the options for each of the objectives. So, for objective $j$ we have $s_{1j}, \ldots, s_{3j}$.

• Then the score associated with each option may be got by the appropriate sum, SW.
Pairwise Comparisons

• The key to AHP is to consider pairwise comparisons of objectives, and of options.

• While this may result (theoretically) in inconsistencies, these can be checked later. (E.g. if A>B and B>C then A should be > C, but this is not a constraint here.)

• Firstly we will look at objectives.
In Practice …

• The objectives are compared with each other. Here these are Salary through to Family. Entries for ‘more important’ items go in first.

• The entry in the matrix $a_{ij}$ takes values with the meanings:
  - i ‘as important’ as j => $a_{ij} = 1$
  - i ‘strongly more important than’ j => $a_{ij} = 5$
  - i ‘absolutely more important than’ j => $a_{ij} = 9$
Complete the Matrix

• Then the rest of the matrix is completed.
• The diagonal takes values $= 1$.
• The other off diagonal elements are completed using the constraint that $a_{ij} = 1/a_{ji}$
• For example,
  – suppose Salary strongly (5) more important than City, very slightly (2) more than Work and somewhat (4) more than Family;
  – Work very slightly (2) than City, and (2) than Family.
  – Family very slightly (2) than City.
Normalise

• Now, normalise this matrix so that the columns sum to one.
• This is done by dividing each entry by the total in the column.
• This is called $A_{\text{norm}}$
• The weight vector, $W$, is then the average of each row.

$$W^T = [0.5115 \quad 0.0986 \quad 0.2433 \quad 0.1466]$$
Repeat with options

• Now, for the first objective, Salary, compare the options. This matrix is called $C_1$.
• The numbers used are as before, and $C_{1\text{norm}}$ is generated in the same fashion.
• $S_1$ is then the vector of the scores given by averaging each row.
Details

\[ C_1 = \begin{bmatrix} 1 & 2 & 4 \\ 0.5 & 1 & 2 \\ 0.25 & 0.5 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 0.5 & 0.33 \\ 2 & 1 & 0.33 \\ 3 & 3 & 1 \end{bmatrix} \]

\[ C_3 = \begin{bmatrix} 1 & 0.14 & 0.33 \\ 7 & 1 & 3 \\ 3 & 0.33 & 1 \end{bmatrix} \quad C_4 = \begin{bmatrix} 1 & 0.25 & 0.14 \\ 4 & 1 & 0.5 \\ 7 & 2 & 1 \end{bmatrix} \]

\[ S = \begin{bmatrix} 0.5714 & 0.1593 & 0.0882 & 0.0824 \\ 0.2857 & 0.2519 & 0.6687 & 0.3151 \\ 0.1429 & 0.5889 & 0.2431 & 0.6025 \end{bmatrix} \]
Scores and Consistency

- By calculation, show that Job 1 scores 0.34, Job 2 0.38 and Job 3 0.28.
- It is clear through the weight and scores matrices how each objective influences these, and how options differ.
- Aw compared to w gives a check for consistency. The average of the elementwise ratios – n divided by (n-1) is denoted CI.
- This should be small (for n=4 <0.09, for n=5 <0.11, ...) Small comes from 10% of the index from a ‘random’ comparison matrix.
Exercise 1

• In University, promotion is due to research (R), teaching (T) and service (S). RvsT is 3, RvsS is 7 and TvsS is 5.

• Two lecturers V and W are compared on R, S and T.
  – On T, V beats W (4).
  – On R, W beats V (3).
  – On S, V beats W (6).

• Which lecturer is ‘better’? Check the consistency matrix.
Goal Programming

• A company may wish to meet a number of objectives.
• Where an ordering of these goals exists, then goal programming can be used.
• Typically, the problem may be set up as a linear program, with multiple objectives.
• The problem is to get as close to a specified value on each objective. It may be impossible to reach all.
Example – advertising impact

Exposure (in millions) of 3 groups to ads during different shows

<table>
<thead>
<tr>
<th></th>
<th>Sports ad</th>
<th>Game show ad</th>
<th>News show ad</th>
<th>Sitcom ad</th>
<th>Drama ad</th>
<th>Soap opera ad</th>
</tr>
</thead>
<tbody>
<tr>
<td>High-income men</td>
<td>7</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>High-income women</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>Low-income people</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>Cost/unit</td>
<td>120</td>
<td>40</td>
<td>50</td>
<td>40</td>
<td>60</td>
<td>40</td>
</tr>
</tbody>
</table>
Example

• We have a budget of €750k
• We need at least 2 sports ads, news ads and drama ads each
• We can have at most 10 of each type of ad
Example

• We have three goals, in order of priority:
  – G1: Get at least 65M HIM exposures
  – G2: At least 72M HIW exposures
  – G3: At least 70M LIP exposures.

• Use LP to see if we can achieve G1?
Strategy

• Run Solver to reach G1.
  – This involves relaxing G2 and G3 constraints.
• Run Solver from this solution with a G1 constraint to reach ‘meet’ G2.
• Given the solution or ‘partial’ at this step, run Solver to ‘meet’ G3 with a G1 and G2 constraint.
Exercise 2

- A company has three products, P1, P2 and P3. This month, demand will be 400 of P1, 500 of P2 and 300 of P3. The company has €17000 for orders and 370sqm of space. Breaching budget costs €1 per €1, and additional space can be obtained for €100 per sq m.

- Costs: P1: 0.6m², €20, stockout €16. P2: 0.5m², €18, stockout €10. P3: 0.4m², €16, stockout €8.

- Objectives;
  - G1: Spend at most 17k.
  - G2: Use at most 370 sqm.
Summary

• Sometimes multiple criteria must be considered when making decisions.
• Pareto TOC, AHP and goal programming are three useful methods, depending on the context.
• You should understand the methods, and (following the lab) know how to implement them in Excel.