Expected Value of a Binomial Distribution

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Recall that we say a random variable \( X \sim \text{Binom}(n, \pi) \) follows a binomial distribution if \( n \) independent trials occur, with a constant probability of success \( \mathbb{P}(\text{Success}) = \pi \), and \( X \) corresponds to the total number of observed successes. In particular, then

\[
\mathbb{P}(X = x) = \binom{n}{x} \pi^x (1 - \pi)^{n-x}, \quad x = 0, \ldots, n.
\]

Then the expected value of \( X \), \( \mathbb{E}[X] \), is found to be:

\[
\mathbb{E}[X] = \sum_{x=0}^{n} x \mathbb{P}(x)
= \sum_{x=1}^{n} x \binom{n}{x} \pi^x (1 - \pi)^{n-x}
= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \pi^x (1 - \pi)^{n-x}
= n\pi \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} \pi^{x-1} (1 - \pi)^{n-x}
= n\pi \sum_{y=0}^{n-1} \frac{(n-1)!}{y!(n-1-y)!} \pi^{y} (1 - \pi)^{n-1-y} \text{ Binom}(n-1, \pi)
= n\pi.
\]

Here we have substituted \( y = x - 1 \), and, noticing that, for the problem at hand \( Y \sim \text{Binom}(n - 1, \pi) \), made use of the fact that \( \sum_{y=0}^{n-1} \mathbb{P}(Y = y) = 1 \).

*Based extensively on material previously taught by Eamonn Mullins.
Variance of a Binomial Distribution

Next we calculate the variance, $\text{Var}[X]$. We will do so in a somewhat indirect way:

$$E[X(X - 1)] = \sum_{x=0}^{n} x(x - 1)\mathbb{P}(x)$$

$$= \sum_{x=0}^{n} x(x - 1) \binom{n}{x} \pi^x (1 - \pi)^{n-x}$$

$$= \sum_{x=2}^{n} x(x - 1) \frac{n!}{(x - 2)!(n - x)!} \pi^x (1 - \pi)^{n-x}$$

$$= n(n - 1)\pi^2 \sum_{x=2}^{n} \frac{(n - 2)!}{(x - 2)!(n - x)!} \pi^{x-2} (1 - \pi)^{n-x}$$

$$= n(n - 1)\pi^2 \sum_{y=0}^{n-2} \frac{(n - 2)!}{y!(n - 2 - y)!} \pi^y (1 - \pi)^{n-2-y}$$

$$= n(n - 1)\pi^2 \text{Binom}(n - 2, \pi)$$

This time we have substituted $y = x - 2$, and, again noticed that one of the terms on the right involves the summation of a probability distribution over its sample space.

Recall that an alternative form for the variance is $\text{Var}[X] = E[X^2] - E[X]^2$. Then we can write:

$$\text{Var}[X] = E[X^2] - E[X]^2$$


$$= E[X(X - 1)]$$

$$= n(n - 1)\pi^2 + n\pi - n^2\pi^2$$

$$= n\pi - n\pi^2$$

$$= n\pi(1 - \pi).$$