Probabilities are a mathematical way to describe an uncertain outcome. For example, suppose a physicist disintegrates 10,000 atoms of an element $A$, and records the outcome each time. Two outcomes are observed:

i). $A \mapsto B + C$ 3,100 times.

ii). $A \mapsto D + E$ 6,900 times.

Then the model for a disintegration of a single atom will be $\mathbb{P}(i) = 0.31; \mathbb{P}(ii) = 0.69$.

In general, we assign to an outcome $A$ a probability $\mathbb{P}(A)$ equal to the relative frequency $r$ that outcome $A$ would appear if an experiment were to replicated many ($N$) times. In particular,

\[
\begin{align*}
\mathbb{P}(A) &= r/N \\
0 &\leq \mathbb{P}(A) \leq 1 \\
\sum_i \mathbb{P}(X_i) &= 1
\end{align*}
\]

where $i$ labels all possible outcomes.

Alternatively, we can also assign probabilities based on our knowledge of physical properties, rather than from conducting numerous experiments. For example, if a coin is fair, then a natural model would be

\[
\begin{align*}
\mathbb{P}(\text{Heads}) &= 1/2 \\
\mathbb{P}(\text{Tails}) &= 1/2.
\end{align*}
\]

In this way, each outcome is equally likely.

*Based extensively on material previously taught by Eamonn Mullins.*
Another obvious example is a fair six sided die:

\[ P(1) = P(2) = \ldots = P(6) = 1/6. \]

We call the set of all possible outcomes the sample space. An event is then any subset of outcomes which we are interested in. An event can then be a single sample point, e.g. “3”, or a collection of points, e.g., “3 or 5,” or “> 2.”

The assignment of probabilities on this basis leads to the following definition for the probability of an event \( A \):

\[ P(A) = \frac{\text{Number of possible outcomes corresponding to } A}{\text{Total number of possible outcomes}}, \]

if the sample points are all considered to be equally likely\(^1\).

**Example 1**

Roll two fair six sided dice and list the resultant face values. The sample space for this example is:

\[
\begin{array}{ccccccc}
1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 \\
2,1 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 \\
3,1 & 3,2 & 3,3 & 3,4 & 3,5 & 3,6 \\
4,1 & 4,2 & 4,3 & 4,4 & 4,5 & 4,6 \\
5,1 & 5,2 & 5,3 & 5,4 & 5,5 & 5,6 \\
6,1 & 6,2 & 6,3 & 6,4 & 6,5 & 6,6 \\
\end{array}
\]

Since all outcomes are equally likely, we assign each outcome a probability of 1/36. What is \( P(\text{sum of dice} = 7) \)? What is \( P(\text{sum of dice} = 9) \)? What is \( P(\text{sum of dice} = 7 \text{ or } 9) \)?

**Mutually Exclusive Events**

If two events \( A \) and \( B \) are mutually exclusive events, then the probability of observing \( A \) or \( B \) (sometimes written as \( P(A \lor B) \)) is the sum of the probabilities:

\[ P(A \text{ or } B) = P(A) + P(B). \]

For example, in Exercise 1, we saw that \( P(\text{sum of dice} = 7 \text{ or } 9) = 10/36 = P(7) + P(9) = 6/36 + 4/36. \)

\(^1\)The number of possible outcomes corresponding to \( A \) is also known as the cardinality of the subset \( A \).
**Example 2**

A card is drawn randomly from a standard deck of cards. A standard deck consists of 52 cards, comprising 13 ranks (2 to 10, jack, queen, king and ace) and 4 suits (clubs, diamonds, hearts and spades). Then the probability the card is the ace of hearts, \( P(\text{Ace of hearts}) = \frac{1}{52} \).

The probability the card is an ace is:

\[
P(\text{Ace}) = P(\text{Ace of clubs or Ace of diamonds or Ace of hearts or Ace of spades})
= P(\text{Ace of clubs}) + P(\text{Ace of diamonds}) + P(\text{Ace of hearts}) + P(\text{Ace of spades})
= \frac{1}{52} + \frac{1}{52} + \frac{1}{52} + \frac{1}{52} = \frac{1}{13}.
\]

**Non-Mutually Exclusive Events**

If two events \( A \) and \( B \) are not mutually exclusive, then the probability of observing \( A \) or \( B \) is the sum of the probabilities:

\[
P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B}).
\]

This prevents us from double counting events which can occur for both \( A \) and \( B \). We often denote \( P(\text{A and B}) \) as \( P(\text{A} \land \text{B}) \), or sometimes as \( P(\text{A, B}) \).

Note that if \( A \) and \( B \) are mutually exclusive, then \( P(\text{A and B}) = 0 \), and we get the expression discussed previously. The second expression is thus the more general form.

**Example 3**

What is the probability that a randomly drawn card from a standard deck is an ace or a spade?

\[
P(\text{Ace or Spade}) = \frac{16}{52}
\]

\[
P(\text{Ace}) = \frac{4}{52}
\]

\[
P(\text{Spade}) = \frac{13}{52}
\]

\[
P(\text{Ace and Spade}) = \frac{1}{52}.
\]
Exercise 1

In a delivery of 1,000 screws, 140 were badly threaded (\(BT\)), 3/4 of these also being rusty \((R)\). Two hundred and fifty screws in total were rusty. If a screw is picked at random, what is the probability that it is:

i). badly threaded;

ii). rusty;

iii). rusty and badly threaded;

iv). rusty or badly threaded;

v). rusty and not badly threaded \((BT = \text{not } BT)\);

vi). badly threaded and not rusty.

Independent Events

Two events \(A\) and \(B\) are statistically independent if \(P(A \text{ and } B) = P(A) \times P(B)\).

For example, suppose we toss a fair coin twice. There are four possible outcomes:

<table>
<thead>
<tr>
<th>Possible Outcomes</th>
<th>HH</th>
<th>HT</th>
<th>TH</th>
<th>TT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assigned Probability</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

Note that we are taking the order of events into account here, i.e., observing a head then a tail is different from observing a tail then a head.
We have already seen that, for a single toss, the probability of a head is the same as that for a tail, $P(H) = P(T) = 1/2$. Then $P(H \land T) = 1/2 \times 1/2 = 1/4$.

Independent events don’t necessarily occur from separate experiments. For example, the probability of randomly drawing the ace of spades from a deck of cards is given by

\[
\begin{align*}
P(\text{Ace of Spades}) &= P(\text{Ace} \land \text{Spade}) \\
&= P(\text{Ace}) \times P(\text{Spade}) \\
&= 1/13 \times 1/4 = 1/52.
\end{align*}
\]