ST1251 Second Assessment

To be handed into the School of Mathematics Office by 12 noon on Monday 16th January, 2017. Alternatively, answers may be handed into my office, room LB144, or sent by email to arwhite@tcd.ie

Print your name and student number on the front of your script.

Answer all questions
1. A company wishes to install a new lift in a building where many of their workers are based. Suppose that the weight in kg of a worker in the building is normally distributed, with an average weight of 75kg and standard deviation of 15.

(a) Let $X_1, \ldots, X_n$ denote the weights of $n$ randomly selected workers in the building. Derive the expectation and variance of $Y_n = X_1 + \ldots + X_n$, the total weight of the selected workers. State explicitly any results which you use for your derivations.

(b) The proposed weight tolerance of the new lift is 420 kg; weights higher than this will put the system at risk of breakdown. Based on this, a safety engineer recommends that no more than 5 people should be allowed in the lift at one time. What is the probability that the system would be at risk if 5 workers were to take the lift at the same time, i.e., what is $P(Y_5 > 420)$?

(c) Let $k$ denote the number of times 5 workers could take the lift before the system would be at risk of breakdown. Give an expression for the probability distribution of $k$.

(d) Explain how you would find a maximum value $n$ such that $P(Y_n > 420) < 0.05$. You do not need to explicitly solve this calculation.

2. (a) If $X$ is a Poisson random variable with parameter $\mu$, derive the expectation and variance of $X$.

(b) In a thesis completed some years ago in the Statistics Department, it was reported that an analysis of 1,140 matches played in the English Premier League between 1997 and 2000 showed that the number of goals scored in a match by a home team could be well approximated by a Poisson distribution (process) with parameter $\mu_H = 1.56$ per 90 minutes. (For simplicity, we assume all matches were 90 minutes duration.)

What is the probability that in a randomly selected match the home team scored more than two goals?

(c) The away goals were also found to follow a Poisson distribution, with scoring rate $\mu_A = 1.10$ per 90 minutes, independently of the number of home goals. Given this, and the information in part (b), what is the probability of a 0-0 draw in a randomly selected match?

(d) Given the Poisson model for the number of goals scored by the home team is $\mu_H = 1.56$ per 90 minutes (or equivalently 1.56/90 per minute), show that the time between goals for the team can be modelled by an exponential distribution parameter, with rate parameter $\lambda_H = \mu_H/90$.

(e) If the home team scores, what is the probability that it will do so again within 20 minutes?

(f) Derive the expectation of the time between goals for a team and evaluate this for the home team.
3. (a) If X is random variable with variance \( \text{Var}[X] \) show that

\[
\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\]

Explicitly state any results that you use in your demonstration.

(b) Let \( X \sim \text{Unif}(a, b) \) be a uniformly distributed random variable, with probability density function

\[
f(x) = \frac{1}{b-a}.
\]

Derive the cumulative density function (cdf), expectation and variance of \( X \).

(c) The system \( S \) is composed of two components, \( C_1 \) and \( C_2 \), in series. In other words, \( S \) works only if both components are working. The two components have lifetimes (in hours) \( T_1 \) and \( T_2 \) respectively, and both lifetimes are independently uniformly distributed in the interval \([0, 100]\), such that \( T_1, T_2 \sim \text{Unif}(0, 100) \).

i. What is the probability that \( T_1 \) is greater than 60 hours?

ii. What is the probability that \( T_S \), the lifetime of the system \( S \), is greater than 60 hours?

iii. What is the probability density function (PDF) for \( T_S \)?

iv. Derive \( \mathbb{E}[T_S] \) and \( \text{Var}[T_S] \), the expectation and variance of the lifetime of the system.

(d) Suppose that ten such systems \( S_1, \ldots, S_{10} \) are each run independently in succession, until failure. What is the probability that the lifetime of exactly four systems will exceed 60 hours? Let \( T_{S*} = T_{S_1} + \cdots + T_{S_{10}} \) be the total lifetime of these systems. What is the expected value of \( T_{S*} \)? What is its variance?
4. Experience suggests that a very large class of mathematics students when tested on a standardised test will achieve a mean score of $\mu_1 = 100$ and standard deviation $\sigma_1 = 10$. Assume that scores are normally distributed.

(a) What value can be calculated such that 5% of the group exceed this value?

(b) If a random sample of 25 of the students is taken and tested, what is the probability that the sample average will exceed 105?

(c) Suppose that 20% of the total class (Group 2) are selected at random for special coaching and that the coaching changes their performance such that the mean for this group is $\mu_2 = 110$ and the standard deviation is $\sigma_2 = 8$. Assume that the remainder of the class (Group 1) have the same characteristics as before. What is the probability that $Y$, the score of a randomly selected student from Group 2, exceeds $X$, that from a randomly selected student from Group 1, by more than 10 units?

(d) A student is randomly selected from the class as a whole (after coaching has taken place.) What is the probability that this student will produce a score greater than 120?

(e) Assume that the student did produce a score greater than 120. What is the probability that the student had coaching?