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Cuddling cats[1]

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Abstract

This paper contains the solutions of few exercises of basic category theory.

Contents

1 Preliminaries

This the paper is not self-contained, and the reader is expected to have a basic knowledge of category theory; material such as books or lecture notes are needed in order to find the definitions which are not explicitly given.

I am responsible for all the mistakes contained in this paper, and, on this score, I would appreciate any feedback. Please send any question or comment to the address `bernargi@tcd.ie`.

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1.1 Set theory

We will use the näive definition of set, so as to be not concerned with foundational issues.

- We say that R is a binary relation between A and B if and only if $R \subseteq A \times B$.
- A *function* is a binary relation $R \subset D \times C$ such that the conditions a and b are true:

  a) if $(x, a) \in R$ and $(x, b) \in R$ then $a = b$
  b) if $x \in D$ then $(x, y) \in R$ for some $y \in C$

  The set D is the *domain* (or source) of R and the set D is the codomain (or target) of R. We denote these sets as $\mathit{src}(R)$ and $\mathit{trg}(R)$. A function can be thought of as a *deterministic* relation. Note that what we refer to as "functions" are often called *total* functions.

- Hereafter we will use the symbol $\equiv$ with the meaning "if and only if".
- We will use the infix notation when reasoning about relations; for instance, if R is a relation then

  $$a(R)b \equiv (a, b) \in R$$

  If R is a function then

  $$b = R(a) \equiv a(R)b$$

- The composition of relations $R \subseteq A \times B$ and $R' \subseteq B \times C$ is denoted $R; R'$ and

  $$a(R; R')b \equiv \exists a'.a(R)a' \wedge a'(R')b$$

1.2 Preorders

Definition 1.1. [Preorder]
A binary relation R on a set S is *preorder* if it is

1. reflexive, for every $a \in S$ we have aRa
2. transitive, for every $a, b, c \in S$ we have aRb, bRc implies aRc □

To denote preorders, we will use the symbol $\leq$, possibly with some subscripts, as in $\leq_A$, $\leq_B$; to denote the algebraic structure given by a set and a preorder relation on it, we will use the symbol $\langle A, \leq_A \rangle$.

Definition 1.2. [Poset]
A partially ordered set, or *poset*, $\leq$ is a preorder relation that is anti-symmetric. Formally, if $a \leq b$ and $b \leq a$ then $a = b$. □

Definition 1.3. [Lower, upper bound]
Let $\langle P, \leq_P \rangle$ be a preorder and let $S \subseteq P$. An element $x \in P$ is a *lower bound* of S if $x \leq_P s$ for every $s \in S$. An *upper bound* is defined dually. The set of lower bounds of S is denoted S^ℓ and defined as

$$S^\ell = \{ x \in P \mid \forall s \in S. x \leq_P s \}$$

□

1.3 Monoids

Lemma 1.4. Let $\langle \{ u \}, *, u \rangle$ be a monoid with only the unit element, and let $\langle M, +, e \rangle$ be a monoid. There exists a *unique* monoid homomorphism $f : M \longrightarrow \{ u \}$.

Proof. Consider the constant function $f(x) = u$ for every $x \in M$. We have to prove that

(a) $f(x + y) = f(x) * f(y)$

(b) $f(e) = u$

Point (b) is true by definition of f. Point (a) is true again by definition of f and because

$$f(x + y) = u = u * u = f(x) * f(y)$$

Now we have to prove that if $g : M \longrightarrow \{ u \}$ is a monoid homomorphism then $g(x) = f(x)$ for every $x \in M$. Since g is a monoid homomorphism, g has to be a function; since the codomain of g has only one element it follows that $g(x) = u = f(x)$ for every $x \in M$. □

Lemma 1.5. [Product of monoids]
Let $\langle M_1, \cdot, u_1 \rangle$ and $\langle M_2, +, u_2 \rangle$ be monoids, and let the operation $*$ be defined as

$$(a, b) * (a', b') = (a \cdot a', b + b')$$

1) The structure $\langle M_1 \times M_2, *, (u_1, u_2) \rangle$ is a monoid

2) the function $fst : M_1 \times M_2 \longrightarrow M_1$ defined as $fst((a, b)) = a$ for every $(a, b) \in M_1 \times M_2$ is a monoid homomorphism

3) the function $snd : M_1 \times M_2 \longrightarrow M_2$ defined as $snd((a, b)) = b$ for every $(a, b) \in M_1 \times M_2$ is a monoid homomorphism

Proof. The proof can be found in any book on algebra, and we skip it. □

Lemma 1.6. Let $\langle M_1 \times M_2, *, (u_1, u_2) \rangle$ be the product of the monoids $\langle M_1, \cdot, u_1 \rangle$ and $\langle M_2, +, u_2 \rangle$, and let $\langle M_3, \circ, u_3 \rangle$ be a monoid such that there exists two monoid homomorphism $f : M_3 \longrightarrow M_1$ and $g : M_3 \longrightarrow M_2$. The function $u : M_3 \longrightarrow M_1 \times M_2$ defined as $u(x) = (f(x), g(x))$ for every $x \in M_3$ is a monoid homomorphism.

Proof. We have to prove that u enjoys two properties,

(a) $u(a \circ b) = u(a) * u(b)$

(b) $u(u_3) = (u_1, u_2)$

We prove point (a).

$$
\begin{array}{ll}
& u(a \circ b) = u(a) * u(b) \\
\equiv & \{\text{definition of } u\} \\
& (f(a \circ b),\, g(a \circ b)) = u(a) * u(b) \\
\equiv & \{f,\, g \text{ monoid homomorphism}\} \\
& (f(a) \cdot f(b),\, g(a) + g(b)) = u(a) * u(b) \\
\equiv & \{\text{definition of } u\} \\
& (f(a) \cdot f(b),\, g(a) + g(b)) = (f(a),\, g(a)) * (f(b),\, g(b)) \\
\equiv & \{\text{definition of } *\} \\
& (f(a) \cdot f(b),\, g(a) + g(b)) = (f(a) \cdot f(b),\, g(a) + g(b)) \\
\equiv & \{\text{identity}\} \\
& \mathit{true}
\end{array}
$$

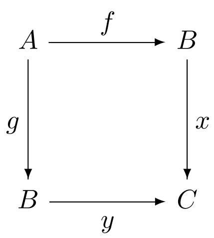
Point (b) follows form the definition of u and the fact that f and g are monoid homomorphism,

$$u(u_3) = (f(u_3),\, g(u_3)) = (u_1,\, u_2)$$

□

1.4 Composition preserves equality

Category theory has one axiom which normally is not explicitly stated ([Awo10, Cro94, Pie91], and that is crucial in our arguments. Let A, B, C and f, g, x, y be objects and arrows as in the diagram below

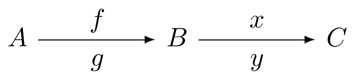


An axiom states that arrow composition preserves equality:

$$\textit{if } f = g \textit{ and } x = y \textit{ then } x \circ f = y \circ g$$

The right-most equality means that the diagram above *commutes.*

In a sense, if $f = g$ and $x = y$ then what we thought of as *two* arrows, turns out to be *one*, that we happen to denote with two different names. For example, under the hypothesis of the axiom, the diagram above collapses to



2 Categories and functors

Exercise 2.1.
no. **1.9** (**1**) in [**Awo10**]

Let **Rel** be the following category: take sets as objects and take binary relations as arrows. The identity arrows on an object A is the identity relation, while the composition of arrows $\circ$ is defined as:

$$R \circ S = S; R$$

(a) Prove that **Rel** is a category

(b) Show also that there is a functor G : **Sets** $\longrightarrow$ **Rel** taking objects to themselves and each function $f : A \longrightarrow B$ to its graph,

$$G(f) = \{\, (a, f(a)) \in A \times B \mid a \in A \,\}$$

(c) Finally, show that there is a functor C : $\mathbf{Rel}^{\mathrm{op}} \longrightarrow$ **Rel** taking each relation $R \subseteq A \times B$ to its converse $R^{\mathrm{op}} \subseteq B \times A$, where,

$$(a,\ b) \in R \text{ if and only if } (b,\ a) \in (R)^{\circ}$$

Proof. (a) We are required to prove that the **Rel** is a category.

We begin by proving the existence of the identity arrows.

First note that for each object A there exsists the identity function ι_A, which we can take as the identity arrow.

We have now to show that the composition of relations ; satifies the axioms of the arrow composition $\circ$.

- Identities:
  Let A be an object of **Rel**; by definition of **Rel**, A is a set, and as suggested in the exercise, we define the identity relation on A as

  $$\iota_A = \{\, (a, a) \mid a \in A \,\}$$

  Note that ι_A is a binary relation, so it is indeed an arrow of **Rel**, and ew take it as the *identity arrow* of the object A.
  Having defined the identity arrows, we have to prove the unit axioms. For every arrow $R : A \longrightarrow B$ we have to show that (1) $R \circ \iota_A = R$ and that (2) $\iota_b \circ R = R$, which by definition of $\circ$ means that we have to prove the following two set equalities

  (1) $\iota_A; R = R$
  (2) $R; \iota_B = R$

  We show point (1); to prove the equality we have to show that $\iota_A; R \subseteq R$ and that $R \subseteq \iota_a; R$. The proof follows.

$$\begin{array}{ll} & a(\iota_A; R)b \\ \equiv & \{\text{Definition of ;}\} \\ & \exists a'.a(\iota_A)a' \wedge a'(R)b \\ \equiv & \{\text{Definition of } \iota_A\} \\ & \exists a'.a = a' \wedge a'(R)b \\ \equiv & \{\text{Meaning of } =\} \\ & (a'(R)b)\{\ a/a'\ \} \\ \equiv & \{\text{Syntactical substitution}\} \\ & a(R)b \end{array}$$

The passages above prove that

$$a(\iota_A; R)b \equiv a(R)b$$

which means that the set inclusions we are after are true.

A simmetrical argument can be used to show that $\iota_b \circ R = R$.

- Composability: We have to show that for any arrows R and S, if $trg(S) = src(R)$ then $R \circ S$ is an arrow of **Rel**, $src(R \circ S) = src(S)$ and $trg(R \circ S) = trg(R)$. These properties follow from the definition of $\circ$ in **Rel** and from the properties of the relational composition ;.
- Associativity:
  Suppose

  $$A \xleftarrow{\ f\ } B \xleftarrow{\ g\ } C \xleftarrow{\ h\ } D$$

  Set theoretically this means that

  $$R \subseteq B \times A, \quad S \subseteq C \times B, \quad T \subseteq D \times C$$

  We have to prove that $R \circ (S \circ T) = (R \circ S) \circ T$. By definition of $\circ$, this amounts is proving $(T; S); R = T; (S; R)$, and to this aim we have to show that two sets inclusions are true. Consider the following logical implications.

  $$\begin{array}{cl} & a[(T;S);R]b \\ \equiv & \{\text{Definition of };\} \\ & \exists c.a(T;S)c \wedge c(R)b \\ \equiv & \{\text{Definition of };\} \\ & \exists c.(\exists d.a(T)d \wedge d(S)c) \wedge c(R)b \\ \equiv & \{\text{Quantifier extrusion}\} \\ & \exists c,d.a(T)d \wedge d(S)c \wedge c(R)b \\ \equiv & \{\text{Quantifier extrusion}\} \\ & \exists d.a(T)d \wedge (\exists c.d(S)c \wedge c(R)b) \\ \equiv & \{\text{Definition of };\} \\ & \exists d.a(T)d \wedge (d(S;R)b) \\ \equiv & \{\text{Definition of };\} \\ & a[T;(S;R)]b \end{array}$$

  In short, the passages above show that $a[T; (S; R)]d \equiv a[(T; S); R]d$, so $(T; S); R \subseteq T; (S; R)$ and $T; (S; R) \subseteq (T; S); R$.

We have shown that the composition of relations ; satisfies the axioms of the arrow composition.

We have proven that **Rel** is a category.

(b) We have to show that G preserves the structure of the category **Sets**. Precisely, we are required to prove the three following properties.

  (i) $f : A \longrightarrow B$ implies $G_a(f) : G_o(A) \longrightarrow G_o(B)$

  (ii) $G_a(\iota_A) = \iota_{G_o(A)}$

  (iii) $G_a(f \circ g) = G_a(f) \circ G_a(g)$

We proceed with the proofs.

(i) Let $f : A \longrightarrow B$ in **Sets**; this means that $f \subseteq A \times B$. By definition

$$G_o(A) = A, \quad G_o(B) = B$$

so $G_o(A)$ and $G_o(B)$ are indeed objects of **Rel**.
We have to prove that $G_a(f)$ is an arrow in **Rel**, that $src(G_a(f)) = A$ and that $trg(G_a(f)) = B$. From the definition of the category **Rel**, it follows that we are required to show that $G_a(f) \subseteq A \times B$. This follows from the very definition of $G_a(f)$ and the fact that $f \subseteq A \times B$.

(ii) Now we want to show that $G_a(\iota_A) = \iota_{G_o(A)}$. Note that from the definition of G it follows $G_o(A) = A$, and so we have $\iota_{G_o(A)} = \iota_A$; this means that we have to prove $G_a(\iota_A) = \iota_A$, that is (1) $\iota_A \subseteq G_a(\iota_A)$ and (2) $G_a(\iota_A) \subseteq \iota_A$.
Consider that by definition of G_a we have

$$G_a(\iota_A) = \{ (a, a) \in A \times A \mid a \in A \}$$

The proof of the set inclusions follows.

$$\begin{array}{cl} & a(G_a(\iota_A))a' \\ \equiv & \{\text{Definition of } G_a(\iota_A)\} \\ & a' = \iota_A(a) \\ \equiv & \{\text{Because } \iota_A \text{ is a function}\} \\ & a(\iota_A)a' \end{array}$$

The proof that $a(\iota_A)a'$ implies $aG_a(\iota_A)a'$ can be obtained reading the passages from the bottom to the top.

(iii) We have left to prove that $G_a(f \circ g) = G_a(f) \circ G_a(g)$. What we really have to prove, thanks to the definition of $\circ$, is that

$$G_a(g; f) = G_a(g); G_a(f)$$

Since the equality above is set-theoretical, we are required to show that $G_a(g; f) \subseteq G_a(g); G_a(f)$ and that $G_a(g); G_a(f) \subseteq G_a(g; f)$. Consider the following passages.

$$\begin{array}{cl} & a[G_a(g; f)]c \\ \equiv & \{\text{Definition of } G_a\} \\ & c = (g; f)(a) \\ \equiv & \{\text{Because } g; f \text{ is a function}\} \\ & a(g; f)c \\ \equiv & \{\text{Definition of ;}\} \\ & \exists b.a(g)b \wedge b(f)c \\ \equiv & \{\text{Because } f \text{ and } g \text{ are functions }\} \\ & \exists b.b = g(a) \wedge c = f(b) \\ \equiv & \{\text{Definition of } G_a\} \\ & \exists b.a(G_a(g))b \wedge b(G_a(f))c \\ \equiv & \text{Definition of ;} \\ & a[G_a(g); G_a(f)]c \end{array}$$

(c) We have to exhibit a suitable C. Instead of showing directly a functor from $\mathbf{Rel}^{\mathrm{op}}$ to $\mathbf{Rel}$, we show a functor from $\mathbf{Rel}^{\mathrm{op}}$ to $(\mathbf{Rel})^{\circ}$, where the objects of $(\mathbf{Rel})^{\circ}$ are sets, and the arrows $(R)^{\circ}$ are defined by the statement

$$(a,\ b) \in R \text{ if and only if } (b,\ a) \in (R)^{\circ}$$

for every binary relation R.

We define C to be the pair (C_a, C_o), where $C_o = \mathcal{I}$ and

$$\begin{array}{lcl} C_a(R^{\mathrm{op}} : A \longrightarrow B) & = & (R)^{\circ} : A \longrightarrow B \\ C_a(R^{\mathrm{op}} \circ S^{\mathrm{op}} : A \longrightarrow C) & = & ((R)^{\circ} : \mathit{src}(R^{\mathrm{op}}) \longrightarrow C) \circ ((S)^{\circ} : A \longrightarrow \mathit{trg}(S^{\mathrm{op}})) \end{array}$$

Now we have to prove the following.

(i) If there exists the arrow $R^{\mathrm{op}} : A \longrightarrow B$ in $\mathbf{Rel}^{\mathrm{op}}$ then there exists $C_a(R^{\mathrm{op}}) : C_o(A) \longrightarrow C_o(B)$ in $(\mathbf{Rel})^{\circ}$

(ii) $C_a(\iota_A) = \iota_{C_o(A)}$

(iii) $C_a(R^{\mathrm{op}} \circ S^{\mathrm{op}}) = C_a(R^{\mathrm{op}}) \circ C_a(S^{\mathrm{op}})$

We give the proofs.

(i) Let $R^{\mathrm{op}} : A \longrightarrow B$ be an arrow of $\mathbf{Rel}^{\mathrm{op}}$. The existence of this arrow means that there exists an arrow $R : B \longrightarrow A$ in $\mathbf{Rel}$, and thus there exists the arrow $(R)^{\circ} : A \longrightarrow B$ in $(\mathbf{Rel})^{\circ}$.
By definition the objects $C_o(A)$ and $C_o(B)$ exists in $(\mathbf{Rel})^{\circ}$, and they are exactly A and B, because the functor C is the identity on objects, thus we have shown that $(R)^{\circ} : C_o(A) \longrightarrow C_o(B)$ exists in $(\mathbf{Rel})^{\circ}$.

(ii) Since $C_o(A) = A$, we are required to prove that $C_a(\iota_A) = \iota_A$. By definition $C_a(\iota_A) = (\iota_A)^{\circ}$, so all we have to show is that $(\iota_A)^{\circ} = \iota_A$; this follows fro mthe definitions of ι_A and $(-)^{\circ}$.

$$\begin{array}{cl} & a((\iota_A)^{\circ})b \\ \equiv & \{\text{Definition of } (-)^{\circ}\} \\ & b(\iota_A)a \\ \equiv & \{\text{Definition of } \iota_A\} \\ & b = a \\ \equiv & \{\text{Definition of } \iota_A\} \\ & a(\iota_A)b \end{array}$$

(iii) Let $R^{\mathrm{op}} : B \longrightarrow C$ and $S^{\mathrm{op}} : A \longrightarrow B$ be two arrows in $\mathbf{Rel}^{\mathrm{op}}$, and consider their composition $R^{\mathrm{op}} \circ S^{\mathrm{op}}$. We have to prove

$$C_a(R^{\mathrm{op}} \circ S^{\mathrm{op}}) = C_a(R^{\mathrm{op}}) \circ C_a(S^{\mathrm{op}})$$

By definition

$$C_a(R^{\mathrm{op}} \circ S^{\mathrm{op}}) = (R)^{\circ} \circ (S)^{\circ}$$

Since the left hand side of the equality above is exactly $C_a(R^{\mathrm{op}}) \circ C_a(S^{\mathrm{op}})$ have nothing more to prove.

We have shown that C is a functor from the category $\mathbf{Rel}^{\mathrm{op}}$ to the category $(\mathbf{Rel})^{\circ}$; our aim, though, is to prove that C is a functor from $\mathbf{Rel}^{\mathrm{op}}$ to $\mathbf{Rel}$. This is true because the category $(\mathbf{Rel})^{\circ}$ equals the category $\mathbf{Rel}$. Indeed, the objects of $\mathbf{Rel}^{\mathrm{op}}$ and $\mathbf{Rel}$ are sets, and for every arrow R in $\mathbf{Rel}$, also $(R)^{\circ}$ is an arrow in $\mathbf{Rel}$., and vice-versa.

□

2.1 Categories and preorders

Hereafter, we use the symbol **Pre** to denote the category of preorders, and the symbol **Cat** the denote the category of *small* categories.

Exercise 2.2.

We can read in pag. 9 [Awo10] the following sentence:

> [...], any category with at most one arrow between any two objects determines a preorder.

I do not understand why the hypothesis *at most one arrow between any two objects* is needed. For instance, below there is a simple of a category with more than one arrow between its objects that determies a preorder. For a more general argument see 2.8.

What the sentence above means is that if the hypothesis is true, than the preorder determied by the category is *isomorphic* to the category itself.

Example 2.3. Consider the finite category:

$$A \overset{f}{\underset{g}{\rightrightarrows}} B$$

Now we define $\leq$ by the statement

$$\text{A} \longrightarrow \text{B if and only if } A \leq B.$$

A more precise statement is

$$(\text{there exists } f : A \longrightarrow B) \text{ if and only if } A \leq B.$$

We apply the latter to the category above, so we get the relation

$$\leq = \{\ (A, A), (B, B), (A, B)\ \}$$

Plainly, the relation $\leq$ is a preorder, and the category above has more than one arrow from A to B. $\square$

Lemma 2.4. Let $\langle\, A, \leq_A \,\rangle$ be a preorder, and let $\leq_{\mathbf{A}}$ have, as objects, the elements of A, and for every pair $(a, b) \in \leq_A$ an arrow $(a, b) : a \longrightarrow b$. Given any arrow (a, b) let $src((a, b)) = a$, $trg((a, b)) = b$; moreover let $\circ$ be defined as follows

$$(a, b) \circ (b, c) = (a, c)$$

The symbol $\leq_{\mathbf{A}}$ denotes a category.

Proof. All the axioms derives in a straightforward way from the properties of the preorders. We prove two of them.

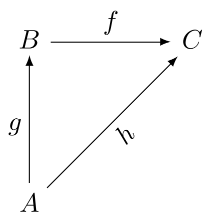
Existence of identities
Let a be an object of $\leq_{\mathbf{A}}$; by definition a is an element of A, so $(a, a) \in \leq_A$ because of the reflexivity of the relation. The identity arrow on the object a is $\iota_a = (a, a)$, and the previous argument shows the existence of an identity function for every object of the category.

Existence composed arrows
If (a, b) and (b, c) are arrows of $\leq_{\mathbf{A}}$ then also their composition, $(a, b) \circ (b, c)$ is an arrow, because by definition it equals (a, c) and because of the transitivity of $\leq_A$. $\square$

Proposition 2.5. [Arrows are unique]
Let $\leq_{\mathbf{P}}$ be the category that arises from the preorder $\langle P, \leq_P \rangle$, and let A and B be two objects of $\leq_{\mathbf{P}}$. There is at most one arrow between any two objects.

Proof. We have to prove that is $f : A \longrightarrow B$ and $g : A \longrightarrow B$ are two arrows in $\leq_{\mathbf{P}}$, then $f = g$. By construction of $\leq_{\mathbf{P}}$, the arrow f is the pair $(A,\, B)$ and the arrow g is the pair $(A,\, B)$. It follows that $f = g$. □

Corollary 2.6. *Let $\leq_{\mathbf{P}}$ be the category that arises from the preorder $\langle P, \leq_P \rangle$. Let f, g, h be arrows in $\leq_{\mathbf{P}}$ as in the following diagram.*



The diagram commutes, that is $f \circ g = h$.

Proof. We know that $f \circ g : A \longrightarrow C$, so Proposition 2.5 implies $f \circ g = h$. □

Lemma 2.7. Let h be a monotone function from the preorder $\langle A, \leq_A \rangle$ to the preorder $\langle B, \leq_B \rangle$, that is $h : \langle A, \leq_A \rangle \longrightarrow \langle B, \leq_B \rangle$ in **Pre**[1].

Define $\hat{h}$ as the pair $(\hat{h})_o$, and $(\hat{h})_a$, where

- $(\hat{h})_o = h$
- $(\hat{h})_a((a, b)) = (h(a), h(b))$

Moreover let $src(\hat{h}) = \leq_{\mathbf{A}}$ and $trg(\hat{h}) = \leq_{\mathbf{B}}$. In the category **Cat** there exists the arrow $\hat{h} : \leq_{\mathbf{A}} \longrightarrow \leq_{\mathbf{B}}$.

Proof. We have to prove that $\hat{h}$ enjoys the three properties of functors.

(1) If $(a, b) : a \longrightarrow b$ is an arrow in $\leq_{\mathbf{A}}$ then $\hat{h}_a((a, b) : a \longrightarrow b) = (c, d) : \hat{h}_o(a) \longrightarrow \hat{h}_o(b)$ is an arrow in $\leq_{\mathbf{B}}$

(2) $\hat{h}_a(\iota_a) = \iota_{\hat{h}_o(a)}$

(3) $\hat{h}_a((a, b) \circ (c, b)) = \hat{h}_a((a, b)) \circ \hat{h}_a((b, c))$

We prove them one by one.

(1)

$$
\begin{aligned}
& (a, b) : a \longrightarrow b \\
\equiv\ & \{\text{By definition of } \leq_{\mathbf{A}}\} \\
& (a, b) \in \leq_A \\
\Rightarrow\ & \{\text{Because } h \text{ is a monotone function}\} \\
& (h(a), h(b)) \in \leq_B \\
\equiv\ & \{\text{By definition of } \leq_{\mathbf{B}}\} \\
& (h(a), h(b)) \in \mathrm{Hom}_{\leq_{\mathbf{B}}}(h(a),\, h(b)) \\
\equiv\ & \{\text{By definition of } \hat{h}_o\} \\
& (h(a), h(b)) \in \mathrm{Hom}_{\leq_{\mathbf{B}}}(\hat{h}_o(a),\, \hat{h}_o(b))
\end{aligned}
$$

[1]and $h : A \longrightarrow B$ in **Sets**.

(2)

$$
\begin{aligned}
& \hat{h}_a(\iota_a) = \iota_{\hat{h}_o(a)} \\
\equiv\ & \{\text{By definition of identity arrow in } \leq_{\mathbf{A}}\} \\
& \hat{h}_a((a,a)) = \iota_{\hat{h}_o(a)} \\
\equiv\ & \{\text{By definition of } \hat{h} \text{ and } h_a((a,b))\} \\
& (h(a), h(a)) = \iota_{\hat{h}_o(a)} \\
\equiv\ & \{\text{By definition of } \hat{h}_o\} \\
& (h(a), h(a)) = \iota_{h(a)} \\
\equiv\ & \{\text{By definition of identity arrow in } \leq_{\mathbf{B}}\} \\
& (h(a), h(a)) = (h(a), h(a)) \\
\equiv\ & \{\text{By identity}\} \\
& \mathit{true}
\end{aligned}
$$

(3)

$$
\begin{aligned}
& \hat{h}_a((a,b) \circ (b,c)) = \hat{h}_a((a,b)) \circ \hat{h}_a((b,c)) \\
\equiv\ & \{\text{By definition of } \circ\} \\
& \hat{h}_a((a,c)) = (h(a), h(b)) \circ (h(b), h(c)) \\
\equiv\ & \{\text{By definition of } \hat{h}_a\} \\
& (h(a), h(c)) = (h(a), h(b)) \circ (h(b), h(c)) \\
\equiv\ & \{\text{By definition of } \circ\} \\
& (h(a), h(c)) = (h(a), h(c)) \\
\equiv\ & \{\text{Identity}\} \\
& \mathit{true}
\end{aligned}
$$

□

Exercise 2.8.
no. **1** (**8**) in [**Awo10**]

Any category $\mathbf{C}$ determines a preorder $P(\mathbf{C})$ by defining a binary relation $\leq$ on the objects by

$$A \leq B \text{ if and only if } A \longrightarrow B$$

(a) Show that P determines a functor from categories to preorders, by defining its effect on functors between categories and checking the required conditions

(b) Show that P is a one-sided inverse to the evident inclusion functor of preorders into categories

Proof. The solution of the exercise is divided in two parts.

(a) We begin by defining the effect of the funtor P on the objects of a category $\mathbf{C}$. Let

$$\leq_{\mathbf{C}} = \{\, (A, B) \mid \mathrm{Hom}_{\mathbf{C}}(A, B) \neq \emptyset \,\}$$

The set $\leq_{\mathbf{C}}$ is a preorder relation. Indeed, the existence of the identity arrows for each object ensure that $\mathrm{Hom}_{\mathbf{C}}(A, A)$ is not empty, and so $\leq_{\mathbf{C}}$ is reflexive. The transitivity of $\leq_{\mathbf{C}}$ follows form the definition of arrow composition. Let $A \leq_{\mathbf{C}} B$ and $B \leq_{\mathbf{C}} C$, then $\mathrm{Hom}_{\mathbf{C}}(A, B) \neq \emptyset$ and $\mathrm{Hom}_{\mathbf{C}}(B, C) \neq \emptyset$, therefore we can pick a $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ and an arrow

g in $\mathrm{Hom}_{\mathbf{C}}(B, C)$, so as to compose them $g \circ f$. The axioms of categories ensure that the composed arrow exists in $\mathbf{C}$, so $\mathrm{Hom}_{\mathbf{C}}(A, C) \neq \emptyset$, and so by definition we have $A \leq_{\mathbf{C}} C$. Note also that we have used no particular assumption on $\mathbf{C}$, thus $\leq_{\mathbf{C}}$ is defined for every category.

We have proven that the algebraic structure $\langle\, obj\mathbf{C},\ \leq_{\mathbf{C}} \rangle$ is a preordered set. We are ready to define the functor P:

$$P_o(\mathbf{C}) = \langle\, obj\mathbf{C},\, \leq_{\mathbf{C}} \rangle$$
$$P_a(F : \mathbf{C} \longrightarrow \mathbf{D}) = F_o : \langle\, obj\mathbf{C},\, \leq_{\mathbf{C}} \rangle \longrightarrow \langle\, obj\mathbf{D},\, \leq_{\mathbf{D}} \rangle$$

We are required to prove that P is a functor from **Cat** to **Pre**, that is we have to show the following facts.

(1) for every functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ in **Cat** the arrow $P(F) : P(\mathbf{C}) \longrightarrow P(\mathbf{D})$ exists in the category **Pre**

(2) $P_a(\iota_{\mathbf{C}}) = \iota_{P_o(\mathbf{C})}$

(3) $P_a(F \circ G) = P_a(F) \circ P_a(G)$

We give the proofs.

(1) Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be an arrow in **Cat**. We are required to show that $P_a(F)$ is a preorder homomorphism, that is for every A, B objects of $\mathbf{C}$, if $A \leq_{\mathbf{C}} B$ then $(P_a(F))(A) \leq_{\mathbf{D}} (P_a(F))(B)$.
By definition $(P_a(F))(A) = F_o(A)$ for every object A, so we have to show that for every A, B objects of $\mathbf{C}$, if $A \leq_{\mathbf{C}} B$ then $F_o(A) \leq_{\mathbf{D}} F_o(B)$.

$$\begin{array}{ll} & A \leq_{\mathbf{C}} B \\ \equiv & \{\text{Definition of } \leq_{\mathbf{C}}\} \\ & \exists f. f \in \mathrm{Hom}_{\mathbf{C}}(A,\, B) \\ \Rightarrow & \{\text{Functoriality of } F\} \\ & \exists F_a(f). F_a(f) \in \mathrm{Hom}_{\mathbf{D}}(F_o(A),\, F_o(B)) \\ \equiv & \{\text{Definition of } \leq_{\mathbf{D}}\} \\ & F_o(A) \leq_{\mathbf{D}} F_o(B) \end{array}$$

(2) We prove the second point: $P_a(\iota_{\mathbf{C}}) = \iota_{P_o(\mathbf{C})}$. By definition $\iota_{P_o(\mathbf{C})} = \iota_{\langle\, obj\mathbf{C}, \leq_{\mathbf{C}} \rangle}$, so we have to show the equality

$$(\iota_{\mathbf{C}})_o = \iota_{\langle\, obj\mathbf{C}, \leq_{\mathbf{C}} \rangle}$$

Note that the arrow $\iota_{\langle\, obj\mathbf{C}, \leq_{\mathbf{C}} \rangle}$ is the identity homomorphism on the preorder $\langle\, obj\mathbf{C}, \leq_{\mathbf{C}} \rangle$, that is

$$A(\iota_{\langle\, obj\mathbf{C}, \leq_{\mathbf{C}} \rangle})B \equiv A = B \tag{1}$$

Now we are ready to prove the result.

$$\begin{array}{ll} & A(\iota_{\mathbf{C}})_o B \\ \equiv & \{\text{Identity}\} \\ & A = B \\ \equiv & \{\text{Equation (1)}\} \\ & A(\iota_{\langle\, obj\mathbf{C}, \leq_{\mathbf{C}} \rangle})B \end{array}$$

(3) Let $F : \mathbf{B} \longrightarrow \mathbf{C}$ and $G : \mathbf{A} \longrightarrow \mathbf{B}$ two functors; we have to prove that $P_a(F \circ G) = P_a(F) \circ P_a(G)$.

We already know that $P_a(F \circ G)$ is a function, because we have proven it is a monotone one.

$$\begin{array}{ll} & A(P_a(F \circ G))C \\ \equiv & \{\text{Definition of } P\} \\ & A(F \circ G)_o C \\ \equiv & \{\text{Definition of funcotr composition}\} \\ & A(F_o \circ G_o)C \\ \equiv & \{\text{Functoriality}\} \\ & \exists B . B = G_o(A) \wedge C = F_o(B) \\ \equiv & \{\text{Definition of } P\} \\ & \exists B . B = (P_a(G))(A) \wedge C = (P_a(F))(B) \\ \equiv & \{\text{Composition of monotone functions}\} \\ & A(P_a(G) \circ P_a(F))C \end{array}$$

(b) For the time being, we limit ourselves to the understanding of which functor is the “evident inclusion functor of preorders into categories”. In particular, we define the functor $I : \mathbf{Pre} \longrightarrow \mathbf{Cat}$, which maps each preorder to a category, and each monotone function to a functor.

Let I be defined by the following pair

$$\begin{array}{l} I_o(\langle A, \leq_A \rangle) = \leq_{\mathbf{A}} \\ I_a(h : \langle A, \leq_A \rangle \longrightarrow \langle B, \leq_B \rangle) = \hat{h} \end{array}$$

We have to prove that I is a functor; that is

(1) If $h : \langle A, \leq_A \rangle \longrightarrow \langle B, \leq_B \rangle$ is an arrow in **Pre** then $I_a(h) : I_o(\langle A, \leq_A \rangle) \longrightarrow I_o(\langle B, \leq_B \rangle)$ is an arrow in **Cat**

(2) $I_a(\iota_{\langle A, \leq_A \rangle}) = \iota_{I_o(\langle A, \leq_A \rangle)}$

(3) $I_a(h \circ g) = I_a(h) \circ I_a(g)$

We prove the three properties stated above.

(1) Let $h : \langle A, \leq_A \rangle \longrightarrow \langle B, \leq_B \rangle$ in **Pre**; we have to show that $I_a(h) : I_o(\langle A, \leq_A \rangle) \longrightarrow I_o(\langle B, \leq_B \rangle)$ is an arrow in **Cat**. In view of Lemma 2.4 we know that $I_o(\langle A, \leq_A \rangle)$ and $I_o(\langle B, \leq_B \rangle)$ are objects of **Cat**, namely $\leq_{\mathbf{A}}$ and $\leq_{\mathbf{B}}$.

By definition we know that $I_a(h) = \hat{h}$, thus, to complete the proof of (1), we are required to show that $\hat{h}$ is a functor from $\leq_{\mathbf{A}}$ to $\mathbf{leq_B}$. This follows from Lemma (2.7).

(2) We have to show that $I_a(\iota_{\langle A, \leq_A \rangle}) = \iota_{I_o(\langle A, \leq_A \rangle)}$. First of all, we use the definitions to show the equality that we will actually prove.

$$\begin{array}{ll} & I_a(\iota_{\langle A, \leq_A \rangle}) = \iota_{I_o(\langle A, \leq_A \rangle)} \\ \equiv & \{\text{By definition of } I_o\} \\ & I_a(\iota_{\langle A, \leq_A \rangle}) = \iota_{\leq_{\mathbf{A}}} \\ \equiv & \{\text{By definition of } I_a\} \\ & \widehat{\iota_{\langle A, \leq_A \rangle}} = \iota_{\leq_{\mathbf{A}}} \end{array}$$

On both sides of the last equality there are functors, so we have to prove *two* equalities, (1) that the functors act in the same fashion on the objects, and (2) that the functors act in the same fashion on the arrows. Formally, we are required to show

(a) $b = (\widehat{\iota_{\langle A, \leq_A \rangle}})_o(a) \equiv b = (\iota_{\leq_{\mathbf{A}}})_o(a)$

(b) $(a, b) = (\widehat{\iota_{\langle A, \leq_A \rangle}})_a((c, d)) \equiv (a, b) = (\iota_{\leq_{\mathbf{A}}})_a((c, d))$

We prove (a).

$$
\begin{array}{cl}
 & b = (\widehat{\iota_{\langle A, \leq_A \rangle}})_o(a) \\
\equiv & \{\text{By definition of } \widehat{\iota_{\langle A, \leq_A \rangle}})_o\} \\
 & b = \iota_{\langle A, \leq_A \rangle}(a) \\
\equiv & \{\text{Because } \iota_{\langle A, \leq_A \rangle} \text{ is an identity function}\} \\
 & a = b \\
\equiv & \{\text{By definition of identity functor}\} \\
 & b = (\iota_{\leq_{\mathbf{A}}})_o(a)
\end{array}
$$

Now we prove (b).

$$
\begin{array}{cl}
 & (a, b) = (\widehat{\iota_{\langle A, \leq_A \rangle}})_a((c, d)) \\
\equiv & \{\text{By definition of } (\widehat{\iota_{\langle A, \leq_A \rangle}})_a\} \\
 & (a, b) = (\iota_{\langle A, \leq_A \rangle}(c), \iota_{\langle A, \leq_A \rangle}(d)) \\
\equiv & \{\text{Identity function}\} \\
 & a = c \wedge b = d \\
\equiv & \{\text{By definition of identity functor } \iota_{\leq_{\mathbf{A}}}\} \\
 & (a, b) = (\iota_{\leq_{\mathbf{A}}})_a((c, d))
\end{array}
$$

(3) Now we have to prove that I preserves composability of arrows, $I_a(h \circ g) = I_a(h) \circ I_a(g)$. Given the definition of I_a, it suffices to prove that

$$\widehat{h \circ g} = \hat{h} \circ \hat{g}$$

From Lemma 2.7 it follows that both sides in the equality above are functors, thus we have to prove two equalities,

(a) $b = (\widehat{h \circ g})_o(a) \equiv b = (\hat{h} \circ \hat{g})_o(a)$

(b) $(a, b) = (\widehat{h \circ g})_a((c, d)) \equiv b = (\hat{h} \circ \hat{g})_a((c, d))$

We prove (a).

$$
\begin{array}{cl}
 & b = (\widehat{h \circ g})_o(a) \\
\equiv & \{\text{By definition of } \widehat{h \circ g}\} \\
 & b = (h \circ g)(a) \\
\equiv & \{\text{By definition of function composition}\} \\
 & b = h(g(a)) \\
\equiv & \{\text{By definition of } (\hat{-})_o\} \\
 & b = (\hat{h})_o((\hat{g})_o(a)) \\
\equiv & \{\text{By definition of function composition}\} \\
 & b = ((\hat{h})_o \circ (\hat{g})_o)(a) \\
\equiv & \{\text{By definition of functor composition}\} \\
 & b = (\hat{h} \circ \hat{g})_o(a)
\end{array}
$$

We prove (b).

$$
\begin{array}{ll}
& (a,b) = (\widehat{h \circ g})_a((c,d)) \\
\equiv & \{\text{By definition of } (\widehat{-})_a\} \\
& (a,b) = ((h \circ g)(c), (h \circ g)(d)) \\
\equiv & \{\text{By definition of function composition}\} \\
& (a,b) = (h(g(c)), h(g(d))) \\
\equiv & \{\text{By definition of } (\widehat{-})_a\} \\
& (a,b) = (\hat{h})_a((g(c), g(d))) \\
\equiv & \{\text{By definition of } (\widehat{-})_a\} \\
& (a,b) = (\hat{h})_a((\hat{g})_a((c,d))) \\
\equiv & \{\text{By definition of function composition}\} \\
& (a,b) = ((\hat{h})_a \circ (\hat{g})_a)((c,d)) \\
\equiv & \{\text{By definition of functor composition}\} \\
& (a,b) = (\hat{h} \circ \hat{g})_a((c,d))
\end{array}
$$

We have thus shown that I is a functor.

□

3 Inverses, isomorphisms, monomorphisms and epimorphisms

Definition 3.1. [Inverse]
We say that the arrow f is the *inverse* of the arrow g (written $f = g^{-1}$) if and only if $f \circ g = \iota_{src}(g)$ and $g \circ f = \iota_{trg}(g)$. More formally,

$$f = g^{-1} \equiv f \circ g = \iota_{src}(g) \wedge g \circ f = \iota_{trg}(g)$$

□

Exercise 3.2.
no. **1.3** (**5**) in **[Pie91]**

Show that if f^{-1} is the inverse of $f : A \longrightarrow B$, and g^{-1} is the inverse of $g : B \longrightarrow C$, then $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$.

Proof. We have to prove that $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. According to Definition (3.1) we have to show

(1) $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \iota_C$

(2) $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \iota_A$

We begin by proving (1); we do so by using an algebraic style rather than a logical one.

$$
\begin{array}{lll}
& (g \circ f) \circ (f^{-1} \circ g^{-1}) = & \\
= & g \circ f \circ f^{-1} \circ g^{-1} & \text{Associativity} \\
= & g \circ (f \circ f^{-1}) \circ g^{-1} & \text{Associativity} \\
= & g \circ \iota_B \circ g^{-1} & \text{By hypothesis} \\
= & (g \circ \iota_B) \circ g^{-1} & \text{Associativity} \\
= & g \circ g^{-1} & \text{Identity} \\
= & \iota_C & \text{Hypothesis}
\end{array}
$$

Now we prove (2) by using a logical style.

$$
\begin{array}{ll}
 & (f^{-1} \circ g^{-1}) \circ (g \circ f) = \iota_A \\
\equiv & \{\text{Associativity}\} \\
 & f^{-1} \circ g^{-1} \circ g \circ f = \iota_A \\
\equiv & \{\text{Associativity}\} \\
 & f^{-1} \circ (g^{-1} \circ g) \circ f = \iota_A \\
\equiv & \{\text{Hypothesis}\} \\
 & f^{-1} \circ \iota_B \circ f = \iota_A \\
\equiv & \{\text{Associativity}\} \\
 & f^{-1} \circ (\iota_B \circ f) = \iota_A \\
\equiv & \{\text{Identity}\} \\
 & f^{-1} \circ f = \iota_A \\
\equiv & \{\text{Hypothesis}\} \\
 & \mathit{true}
\end{array}
$$

The proof of $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \iota_A$ is analogous. Rationale: if f and g are composable arrows then $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$. □

Definition 3.3. [Monomorphism]
Let f be an arrow in some category. We say that f is a *monomorphism* whenever $f \circ g = f \circ h$ implies $g = h$, for every arrows g, h that compose with f on the right. □

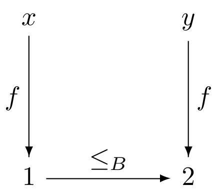
Exercise 3.4.
in pag. 12 [Awo10] and no. **1.2.18** (**3**) in **[Cro94]**

Find a counterexample to the following statement. A monotone function $f : X \longrightarrow Y$ between posets X and Y which is a bijection is necessarily an isomorphism.

Proof. We are free to choose two posets that suit our aim. The simplest case involves two posets $\langle\, A, \leq_A \,\rangle$ and $\langle\, B, \leq_B \,\rangle$ defined as follows.

$$
\begin{array}{llll}
A & = \{\, x, y \,\}, & \leq_A & = \{\, (x,x), (y,y) \,\} \\
B & = \{\, 1, 2 \,\}, & \leq_B & = \{\, (1,1), (2,2), (1,2) \,\}
\end{array}
$$

Now we have to choose a bijection on the underlying sets, which is monotone, but whose inverse do *not* preserve the order structure. This is easy because one poset is discrete, while the other is not. Let $f : A \longrightarrow B$ be defined as $f(x) = 1, f(y) = 2$; its inverse is $f^{-1}(1) = x, f^{-1}(2) = y$. We depict the situation below.



The function f enjoys the following properties

1. it is total

2. it is injective: $\forall x, y \in A. f(x) \leq_B f(y)$ implies $x = y$

3. it is surjective: $\forall y \in B. \exists x \in A. f(x) = y$

4. it is monotone: $\forall x, y \in A. x \leq_A y$ implies $f(x) \leq_B f(y)$

It follows that $f : A \longrightarrow B$ is an iso in the category **Sets**, while $f : \langle A, \leq_A \rangle \longrightarrow \langle B, \leq_B \rangle$ is an arrow in the category **Pre**.

Note now that $1 \leq_B 2$ and that $f^{-1}(1) \not\leq_A f^{-1}(2)$; that is f^{-1} is not monotone, hence f is not an arrow in **Pre**. Since the inverse of a function is unique, the problem does not depend on the f^{-1} we have picked. □

Exercise 3.5.

no. **1.3.10** (**2**) in **[Pie91]**

Show that in any category, if two arrows f and g are both monic than their composition $g \circ f$ is monic. Also, if $f \circ g$ is monic then so is f.

Proof. We have to show that, under the assumptions of the arrows f and g begin monic, the composition $g \circ f$ is monic.

$(g \circ f) \circ x = (g \circ f) \circ y$	By assumption
$g \circ (f \circ x) = g \circ (f \circ y)$	By associativity
$f \circ x = f \circ y$	Because g is monic
$x = y$	Because f is monic

With respect to the second part of the exercise, see part (b) of Exercise (3.10). □

Exercise 3.6.

no. **1.3.10** (**3**) in **[Pie91]**

Dualize the previous exercise: state and prove the analogous proposition for epics. (Be careful on the second part).

Proof. Let be f and g two epimorphisms in a category. We have to prove that $f \circ g$ is an epimorphism.

$x \circ (f \circ g) = y \circ (f \circ g)$	By assumption
$(x \circ f) \circ g = (y \circ f) \circ g$	Associativity
$x \circ f = y \circ f$	Because g is epic
$x = y$	Because f is epic

Now we should prove that if $f \circ g$ is epic then so is g. This is part (c) of Exercise (3.10). □

Exercise 3.7.

no. **1.3.10** (**4**) in **[Pie91]**

no. **2.8** (**3**) in **[Awo10]**

Show that if f is an isomorphism then its inverse f^{-1} is unique.

Proof. Let g be an inverse of the function $f : A \longrightarrow B$; we have to prove that $g = f^{-1}$. The proof relies on Definition 3.1.

$$
\begin{aligned}
& f \circ g = \iota_B \\
\equiv\ & \{\text{By hypothesis and Definition 3.1}\} \\
& f \circ g = f \circ f^{-1} \\
\equiv\ & \{\text{Because of Definition 3.1}\} \\
& f^{-1} \circ (f \circ g) = f^{-1} \circ (f \circ f^{-1}) \\
\equiv\ & \{\text{By associativity}\} \\
& (f^{-1} \circ f) \circ g = (f^{-1} \circ f) \circ f^{-1} \\
\equiv\ & \{\text{Because of Definition 3.1}\} \\
& \iota_A \circ g = \iota_A \circ f^{-1} \\
\equiv\ & \{\text{By unit axiom}\} \\
& g = f^{-1}
\end{aligned}
$$

□

Exercise 3.8.

no. **1.3.10** (**1**) in **[Pie91]**

no. **2.8** (**1**) in **[Awo10]**

Show that a function between sets is an epimorphism if and only if it is surjective. Conclude that isos in **Sets** are exactly the epi-monos.

Proof. The proof is divided in three parts. First we show that if a function f is surjective then f is an epimorphism. Let $f : A \longrightarrow B$ be such that for every $b \in B$ there exists $a \in A$ with $f(a) = b$. Let g, h be two functions composable on the left with f; that is $src(h) = src(g) = B$. We prove that $h \circ f = g \circ f$ implies $h = g$.

Assume $h \circ f = g \circ f$. We have to show that $h(b) = g(b)$ for every $b \in src(h)$. Pick a $b \in src(h)$. Then there exists a $a \in A$ such that $f(a) = b$, because f is surjective. Now note that

$$
\begin{aligned}
h(b) &= h(f(a)) && \text{By construction of } a \text{ and } b \\
&= (h \circ f)(a) && \text{By definition} \\
&= (g \circ f)(a) && \text{By assumption} \\
&= g(f(a)) && \text{By definition} \\
&= g(b) && \text{By construction of } a \text{ and } b
\end{aligned}
$$

Now we give the second part of the proof. We have to show that if $f : A \longrightarrow B$ is an epi then it is surjective. So, assume f to be an epimorphism:

$$\text{if } h \circ f = g \circ f \text{ then } h = g$$

We have to show that for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$. The argument is by contradiction. Suppose that f is *not* surjective. Then there exists $b' \in B$ such that $b' \neq f(a)$ for every $a \in A$. Take an arrow $h : B \longrightarrow C$;

the axioms of $\circ$ ensure that $h \circ f$ is defined. Since in **Sets** functions are total $h(b')$ is defined; fix a $c \in C$ such that $c \neq h(b')$ and define the function

$$g(b) = \begin{cases} c & \text{if } b = b', \\ h(b) & \text{if } b \neq b' \end{cases}$$

Since $b' \notin f(A)$, for every $a \in A$ we have the following

$$(g \circ f)(a) = g(f(a)) = h(f(a)) = (h \circ f)(a)$$

and, therefore, $h = g$ because f is an epimorphism. But this is false, because by construction $g(b') = c \neq h(b')$.

At last, we have the third part of the proof.

Now we have to show that an arrow f is an isomorphism if and only if it is a surjective and injective function (ie. bijection).

If) On the one hand, if f is a bijection then it is total, surjective and injective, therefore it is an arrow, and we can prove that it is an epi (because of surjectivity) and that it is a mono (because of injectivity).

Only if) On the other hand if the arrow f is an isomorphism then f is a monomorphism and an epimorphism (see Exercise 3.12), and therefore surjective and injective.

□

Exercise 3.9.

no. **2.8** (**2**) in **[Awo10]**

Show that in a poset category, all arrows are both monic and epic.

Proof. Let us recall the definition of arrow in a poset category **C**; an arrow f in **C** is a *pair* (a, b) such that $(a, b) \in\leq$, being $\leq$ the preorder relation used to generate **C**.

We prove that any arbitrary arrow is monic. Let f, g, h be arrows in **C** such that $f \circ h = f \circ g$. We are required to show $h = g$.

What does it mean that $f \circ h = f \circ g$? It means that $(src(h), trg(f)) = (src(g), trg(f))$, hence we know $src(h) = src(g)$. Furthermore, since both h and g are composable on the right with f it must be $trg(h) = src(f) = trg(g)$. We have enough information to state that $h = (src(h), src(f)) = (src(g), src(f)) = g$.

The argument to show that any arrow is epic is symmetrical.

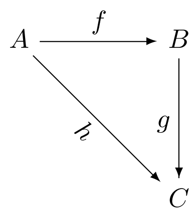
□

Exercise 3.10.

no. **2.8** (**4**) in **[Awo10]**

Point (a) is part of no. **2.5.2** (**2**) in **[Cro94]**

With regard to the commutative triangle,



in any category **C**, show

(a) if f and g are isos (resp. monos, resp. epis), so is h;

(b) if h is monic, so is f

(c) if h is epic, so is g

(d) (by example) if h is monic, g need not be

Proof. Before turning t the proofs, note that the commutativity of the diagram above is algebraically expressed by the equality $h = g \circ f$.

(a) We prove that if f and g are monos, so is h. Assume $h \circ l = h \circ m$ for some suitable l, m, then

$$\begin{array}{ll} & h \circ l = h \circ m \\ \equiv & \{\text{Because of commutativity}\} \\ & (g \circ f) \circ l = (g \circ f) \circ m \\ \equiv & \{\text{By associativity}\} \\ & g \circ (f \circ l) = g \circ (f \circ m) \\ \Rightarrow & \{\text{Because } g \text{ is monic}\} \\ & f \circ l = f \circ m \\ \Rightarrow & \{\text{Because } f \text{ is monic}\} \\ & l = m \end{array}$$

By a similar argument we prove that if f and g are epis, so is h. We use a different style of proof.

$$\begin{array}{cr} l \circ h = m \circ h & \text{Assumption} \\ l \circ (g \circ f) = m \circ (g \circ f) & \text{Because the diagram commutes} \\ (l \circ g) \circ f = (m \circ g) \circ f & \text{Associativity} \\ l \circ g = m \circ g & \text{Because } f \text{ is epi} \\ l = m & \text{Because } g \text{ is epi} \end{array}$$

The two proofs embody the same argument; one given in **C** and the other one in $\mathbf{C}^{\mathrm{op}}$, so the latter proof is the dual of the former one.

Now we prove that if f and g are both isos, so is h. We have to show an arrow h' such that

(1) $h \circ h' = \iota_C$

(2) $h' \circ h = \iota_A$

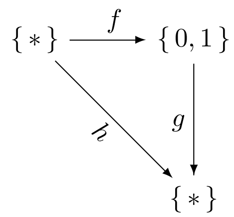
As candidate h' we choose $(g \circ f)^{-1}$. We have to be sure that this arrow exists; in Exercise 3.2 we have seen that it does. Since we already know that $h = g \circ f$, (1) and (2) follow from Exercise 3.2.

(b) If h is monic, so is f.

$$
\begin{array}{ll}
 & f \circ m = f \circ l \\
\equiv & \{\text{Composition preserves equality}\} \\
 & g \circ (f \circ m) = g \circ (f \circ l) \\
\equiv & \{\text{Associativity}\} \\
 & (g \circ f) \circ m = (g \circ f) \circ l \\
\equiv & \{\text{Commutativity diagram}\} \\
 & h \circ m = h \circ l \\
\equiv & \{\text{Because } h \text{ is monic}\} \\
 & m = l
\end{array}
$$

(c) The argument is the one used in the previous case, but this time applied to the arrows in $\mathbf{C}^{\mathrm{op}}$

(d) We give the example in the category **Sets**. We have to show that there is an injective h which equals the composition $f \circ g$ of an injective f and of a function g which is *not* injective.



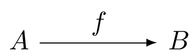
Plainly, the function h is injective, and so is f, whereas g is not, because $g(0) = g(1) = *$ and $0 \neq 1$.

□

Exercise 3.11.
no. **1.3.10** (**6**) in [**Pie91**]

Find a category containing an arrow that is both a monomorphism and an epimorphism, but not an isomorphism.

Proof. Consider the diagram



We have omitted the identity arrows ι_A and ι_B. The arrow f is a monomorphism. This is true because in the category drawn above there is a unique arrow which composes with f on the right, namely ι_B. For the symmetrical reason f is an epimorphism. Indeed, the only arrow which composes with f on the left is ι_A. Note, now, that f cannot be an isomorphism, for there is no arrow form B to A, thus f has no inverse. □

Exercise 3.12.
no. **2.8** (**5**) in [**Awo10**]

Show that the following are equivalent for an arrow

$$f : A \longrightarrow B$$

in any category.

(a) f is an isomorphism

(b) f is both a mono and a split epi

(c) f is both a split mono and an epi

(d) f is both a split mono and a split epi

Proof. In order to solve this exercise we use the standard technique to prove statements equivalent, which consists in showing these implications:

(a) implies (b) implies (c) implies (d) implies (a)

We have to prove four implications.

(1) Suppose f is an isomorphism, then it has an inverse f'.

From the hypothesis it follows that f is a split epi, because it has right inverse, namely f'.

To show that f is a monic, on the contrary, there is little work to be done:

$$\begin{aligned}
f \circ x &= f \circ y && \text{Assumption} \\
f' \circ (f \circ x) &= f' \circ (f \circ y) && \text{Because of the previous eq.} \\
(f' \circ f) \circ x &= (f' \circ f) \circ y && \text{Associativity} \\
\iota_a \circ x &= \iota_a \circ y && \text{By Definition 3.1} \\
x &= y && \text{Identity axioms}
\end{aligned}$$

(2) Let f be a mono and a split epi; we denote the right inverse of f with f', so $f \circ f' = \iota_B$. The fact that f is an epi follows form its being a split epi:

$$\begin{aligned}
x \circ f &= y \circ f && \text{Assumption} \\
(x \circ f) \circ f' &= (y \circ f) \circ f' && \text{Composition preserves equality} \\
x \circ (f \circ f') &= y \circ (f \circ f') && \text{Associativity} \\
x \circ \iota_B &= y \circ \iota_B && \text{By def. of split epi} \\
x &= y && \text{By unit axioms}
\end{aligned}$$

Now we show that f is a split mono. We are required to exhibit a *left* inverse of f.

$$
\begin{array}{ll}
 & f \text{ split epi} \\
\equiv & \{\text{By definition}\} \\
 & \exists f'. f \circ f' = \iota_B \\
\Rightarrow & \{\text{Composition preserves equality}\} \\
 & \exists f'. (f \circ f') \circ f = \iota_B \circ f \\
\equiv & \{\text{By associativity}\} \\
 & \exists f'. f \circ (f' \circ f) = \iota_B \circ f \\
\equiv & \{\text{By unit axioms}\} \\
 & \exists f'. f \circ (f' \circ f) = f \\
\equiv & \{\text{By unit axioms}\} \\
 & \exists f'. f \circ (f' \circ f) = f \circ id_A \\
\Rightarrow & \{f \text{ is monic}\} \\
 & \exists f'. f' \circ f = id_A \\
\equiv & \{\text{By definition}\} \\
 & f \text{ split mono}
\end{array}
$$

(3) Suppose f is a split mono and an epi. We prove that it is a split mono and a split epi. One argument holds by hypothesis, for we supposed f to be split mono.

To show that f is a split epi we have to prove that f has a right inverse. The argument is the dual to the one used in the previous case of the exercise.

$$
\begin{array}{ll}
 & f \text{ split mono} \\
\equiv & \{\text{By definition}\} \\
 & \exists f'. f' \circ f = \iota_B \\
\Rightarrow & \{\text{Composition preserves equality}\} \\
 & \exists f'. f \circ (f' \circ f) = f \circ \iota_A \\
\equiv & \{\text{By associativity}\} \\
 & \exists f'. (f \circ f') \circ f = f \circ \iota_A \\
\equiv & \{\text{By unit axioms}\} \\
 & \exists f'. (f \circ f') \circ f = f \\
\equiv & \{\text{By unit axioms}\} \\
 & \exists f'. (f \circ f') \circ f = id_B \circ f \\
\Rightarrow & \{f \text{ is epic}\} \\
 & \exists f'. f' \circ f = id_A \\
\equiv & \{\text{By definition}\} \\
 & f \text{ split epi}
\end{array}
$$

(4) Let f be both split mono and split epi. The f is an isomorphism. This is true because by hypothesis f has both right and left inverse, and we can prove that they coincide.

□

4 Initial and Terminal objects

Proposition 4.1. [UMP terminal objects] The object $\mathbf{1}$ is terminal if and only if

$$h : A \longrightarrow \mathbf{1} \equiv h = \langle\rangle_A$$

Exercise 4.2.
no. **1.4.6** (**1**) in **[Pie91]**

Show that terminal objects are unique up to isomorphism, that is, that two terminal objects in the same category must be isomorphic. Use duality to obtain a short proof that any two initial objects are isomorphic.

Proof. Consider, in a category $\mathbf{C}$, the following diagram of *two* terminal objects:

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

As usual we have omitted the identity arrows. By definition, for each object X we have the two universal properties

$$\begin{array}{lcl} h : X \longrightarrow A & \equiv & h = \langle\rangle_X \\ h : X \longrightarrow B & \equiv & h = \langle\rangle'_X \end{array}$$

Since $\iota_A : A \longrightarrow A$ we have that $\iota_A = \langle\rangle_A$, and similarly $\iota_B = \langle\rangle'_B$. Note that the arrows f and g compose, and, therefore,

$$f \circ g : B \longrightarrow B, \quad g \circ f : A \longrightarrow A$$

Using the universal properties above we can conclude that $f \circ g = \iota_B$ and $g \circ f = \iota_A$. This means that f is an isomorphism.

The final objects of $\mathbf{C}$ are mapped to inital objects in $\mathbf{C}^{\mathrm{op}}$; this is true because $(-)^{\mathrm{op}}$ reverses the arrows without adding new ones, so the image of each terminal object has exactly one arrow to any object of $\mathbf{C}$.

The fact that $(-)^{\mathrm{op}}$ is a functor can be used to prove that the initial objects in $\mathbf{C}^{\mathrm{op}}$ are isomorphic, because functores preserve isomorphisms. Since any category has its dual, and $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}$, in any category the inital objects are isomorphic. □

Exercise 4.3.
no. **1.4.6** (**3**) in **[Pie91]**

Name a category with no initial objects. Name one with no terminal objects. Name one where the inital and terminal ojects are the same.

Proof. (a) The empty category has no initial objects. We give also a less trivial instance. A category with no initial object is the category given by a monoid with two elements. For instance

$$* \underset{0}{\overset{1}{\rightleftarrows}} *$$

In such a category there is a unique object $*$ and it has two arrows to itself. It follows that each object in such a category as the one above has *more than one* arrow to any object in the category, and is reached by *more than one* arrow by any object in the category; thus it is neither initial not terminal.

(b) The empty category has no terminal objects. Also the category

$$A \xleftarrow{\;g \circ f\;} A \underset{g}{\overset{f}{\rightleftarrows}} B \xrightarrow{\;f \circ g\;} B$$

with A and B *not* isomorphic has no terminal object. In this case each object has three arrows entering in it, namely the identity, f or g, and a suitable composition of f and g.

(c) A suitable category is the following:

$$A \xrightarrow{\iota_A} A$$

As it stands, it is easy to see that A is initial, for there exists a unique arrow $\iota_A : A \longrightarrow A$ to any object. An analogous argument shows that A is terminal.

□

5 Products

Definition 5.1. [Product]
Given a category $\mathbf{C}$, the *product* of two objects A and B is a triple composed by an object $A \times B$, and two *projection arrows* $\pi_A : A \times B \longrightarrow A$, $\pi_B : A \times B \longrightarrow B$, such that for every object C and pair of arrows $f : C \longrightarrow A$, $g : C \longrightarrow B$ there **exists** a **unique** arrow $\langle f, g \rangle : C \longrightarrow A \times B$ such that $\pi_A \circ \langle f, g \rangle = f$ and $\pi_B \circ \langle f, g \rangle = g$. □

Definition 5.2. [Category with products]
We say that a category $\mathbf{C}$ has products if and only if for every object A and B there exists the product of A and B, that is the triple $(A \times B, \pi_A : A \times B \longrightarrow A, \pi_B : A \times B \longrightarrow B)$. □

The universal property of products is

$$h = \langle f, g \rangle \equiv \pi_A \circ h = f \wedge \pi_B \circ h = g$$

Definition 5.3. [Empty product]
In any category $\mathbf{C}$ a terminal object $\mathbf{1}$ is an *empty product.* □

Lemma 5.4. [Fusion law]
Let $\mathbf{C}$ be a category and A, B, C, X three of its object; moreover let the product $A \times B$ be defined. For every arrow $h : X \longrightarrow C$ we have that

$$\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$$

Proof.

$$\begin{array}{ll}
& \langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle \\
\equiv & \{\text{By universal property of products}\} \\
& \pi_A \circ (\langle f, g \rangle \circ h) = f \circ h \wedge \pi_B \circ (\langle f, g \rangle \circ h) = g \circ h \\
\equiv & \{\text{By associativity}\} \\
& (\pi_A \circ \langle f, g \rangle) \circ h = f \circ h \wedge (\pi_B \circ \langle f, g \rangle) \circ h = g \circ h \\
\equiv & \{\text{By definition of projection arrows}\} \\
& f \circ h = f \circ h \wedge g \circ h = g \circ h \\
\equiv & \{\text{Composition preserves equality}\} \\
& \mathit{true} \wedge \mathit{true}
\end{array}$$

□

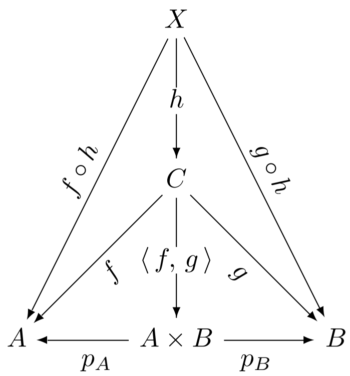
Exercise 5.5.

no. **1.5.6** (**1**) in [**Pie91**]

no. **2** (**2.6.7**) in [**Cro94**]

Show that $\langle f \circ h, g \circ h \rangle = \langle f, g \rangle \circ h$. (Begin by drawing a diagram.)

Proof. We have already proven the result in Lemma 5.4, so we limit ourselves the picture of the diagram at issue.



□

Exercise 5.6.

no. **1.5.6** (**4**) in [**Pie91**]

Let X and Y be objects in a poset P considered as a category. What is a product of X and Y?

Proof. Let $\langle P, \leq_P \rangle$ be a poset. We have seen in Lemma 2.4 that the poset gives rise to a category $\leq_{\mathbf{P}}$. Let X and Y be objects in the category $\leq_{\mathbf{P}}$.

Suppose that the product of X and Y exists; the product is a triple composed by an object $X \times Y$ and two arrows, $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$.

The existence of the projection arrows π_X and π_Y and the construction of $\leq_{\mathbf{P}}$, ensure that $(X \times Y, X) \in \leq_P$ and $(X \times Y, Y) \in \leq_P$; that is $X \times Y$ is a lower bound of $\{ X, Y \}$ (ie. $X \times Y \in \{ X, Y \}^{\ell}$).

Now we prove that $X \times Y$ is the greatest element of $\{ X, Y \}^{\ell}$. Let $C \in setX, Y^{\ell}$; we have to show that $(C, X \times Y) \in \leq_P$.

Since $C \in setX, Y^{\ell}$, there exist two arrows $f : C \longrightarrow X$ and $g : C \longrightarrow Y$, thus by definition of product there exists an arrow $\langle f, g \rangle : C \longrightarrow X \times Y$. This is enough to state that $(C, X \times Y) \in \leq_P$.

We have shown that the object $X \times Y$ is the greatest lower bound of X and Y, that is $X \times Y = X \wedge Y$. □

Definition 5.7. [Product map] If $A \times C$ and $B \times D$ are product objects, then for every pair of arrows $f : A \longrightarrow B$ and $g : C \longrightarrow D$, the *product map* $f \times g : A \times C \longrightarrow B \times D$ is the arrow $\langle f \circ \pi_1, g \circ \pi_2 \rangle$. □

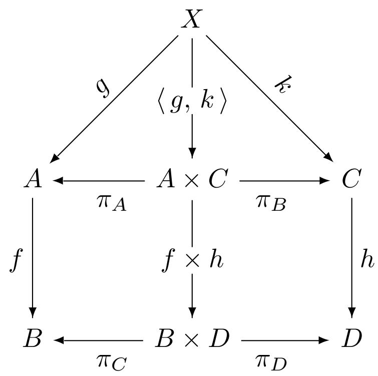
Exercise 5.8.

no. **1.5.6** (**2**) in [**Pie91**]

no. **2** (**2.6.7**) in [**Cro94**]

Show that $(f \times h) \circ \langle g, k \rangle = \langle f \circ g, h \circ k \rangle$.

Proof.



We begin by stating the universal properties of the arrows in the diagram:

$$\langle\, g,\, k \,\rangle = h \quad\equiv\quad \pi_A \circ h = g \wedge \pi_B \circ h = k \tag{2}$$

$$\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle = h' \quad\equiv\quad \pi_D \circ h' = f \circ \pi_A \wedge \pi_C \circ h' = h \circ \pi_B \tag{3}$$

$$\langle\, f \circ g,\, h \circ k \,\rangle = h' \quad\equiv\quad \pi_C \circ h' = f \circ g \wedge \pi_D \circ h' = h \circ k \tag{4}$$

$(f \times h) \circ \langle\, g,\, k \,\rangle = \langle\, f \circ g,\, h \circ k \,\rangle$

$\equiv$ {By Definition 5.7}

$\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle = \langle\, f \circ g,\, h \circ k \,\rangle$

$\equiv$ {Universal property of $\langle\, f \circ g,\, h \circ k \,\rangle$}

$\pi_C \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = f \circ g$

$\wedge$

$\pi_D \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = h \circ k$

$\equiv$ {Universal property of $\langle\, g,\, k \,\rangle$}

$\pi_C \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = f \circ (\pi_A \circ \langle\, g,\, k \,\rangle)$

$\wedge$

$\pi_D \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = h \circ (\pi_B \circ \langle\, g,\, k \,\rangle)$

$\equiv$ {By associativity}

$\pi_C \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = (f \circ \pi_A) \circ \langle\, g,\, k \,\rangle$

$\wedge$

$\pi_D \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = (h \circ \pi_B) \circ \langle\, g,\, k \,\rangle$

$\equiv$ {By universal property $\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle$}

$\pi_C \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = (\pi_D \circ \langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle) \circ \langle\, g,\, k \,\rangle$

$\wedge$

$\pi_D \circ (\langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle \circ \langle\, g,\, k \,\rangle) = (\pi_C \circ \langle\, f \circ \pi_A,\, h \circ \pi_B \,\rangle) \circ \langle\, g,\, k \,\rangle$

$\equiv$ {Composition preserves equality}

$\mathit{true} \wedge \mathit{true}$

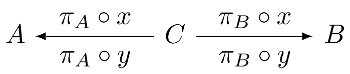
□

Lemma 5.9. Let $A \times B$, π_A, and π_B be a product of two objects A and B; moreover let

$$C \underset{y}{\overset{x}{\rightrightarrows}} A \times B$$

If $\pi_A \circ x = \pi_A \circ y$ and $\pi_B \circ x = \pi_B \circ y$ then $x = y$.

Proof. By hypothesis we have the following diagram



thus, form definition 5.1 it follows that there exists two arrows $\langle\, \pi_A \circ x,\, \pi_B \circ x \,\rangle$ and $\langle\, \pi_A \circ y,\, \pi_B \circ y \,\rangle$ such that

$$
\begin{aligned}
h = \langle\, \pi_A \circ x,\, \pi_B \circ x \,\rangle &\equiv \pi_A \circ h = \pi_A \circ x \wedge \pi_B \circ h = \pi_B \circ x \\
h = \langle\, \pi_A \circ y,\, \pi_B \circ y \,\rangle &\equiv \pi_A \circ h = \pi_A \circ y \wedge \pi_B \circ h = \pi_B \circ y
\end{aligned}
$$

It is straightforward to observe that $x = \langle\, \pi_A \circ x,\, \pi_B \circ x \,\rangle$ and $y = \langle\, \pi_A \circ y,\, \pi_B \circ y \,\rangle$; thanks to the hypothesis we know that $\langle\, \pi_A \circ x,\, \pi_B \circ x \,\rangle = \langle\, \pi_A \circ y,\, \pi_B \circ y \,\rangle$, and we conclude that $x = y$. □

Corollary 5.10.

$$\pi_A \circ x = \pi_A \circ y \wedge \pi_B \circ x = \pi_B \circ y \equiv x = y$$

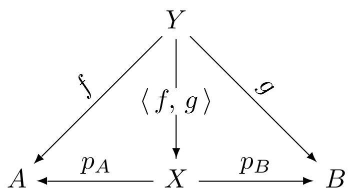
Exercise 5.11.
no. **1.5.2** (−) in **[Pie91]**

Show that any object X with arrows $\pi_A : X \longrightarrow A$ and $\pi_B : X \longrightarrow B$ satisfying the definition of "X is a product of A and B" is isomorphic to $A \times B$. Conversely, show that any object isomorphic to a product object $A \times B$ is product of A and B.

Proof. The exercise has two parts, say (a) and (b).

We begin by solving (a); we have to prove that any two objects X and Y, which are products of A and B, are isomorphic. To this aim is suffices to prove that if X is a product of A and B then it is isomorphic to $A \times B$; formally $X \cong A \times B$. In particular we have to show that two arrows $f : X \longrightarrow A \times B, g : A \times B \longrightarrow X$ exist, such that $f \circ g = \iota_{A \times B}$ and $g \circ f = \iota_X$.

By hypothesis X is a product of A and B, thus by Definition 5.1 we have the diagram



with the universal property

$$h = \langle\, f,\, g \,\rangle \equiv p_A \circ h = f \wedge p_B \circ h = g$$

for any object Y that has two arrows to A and B

We also know that the product object $A \times B$ exists, and so do the two projection arrows $\pi_A : A \times B \longrightarrow A$ and $\pi_B : A \times B \longrightarrow B$.

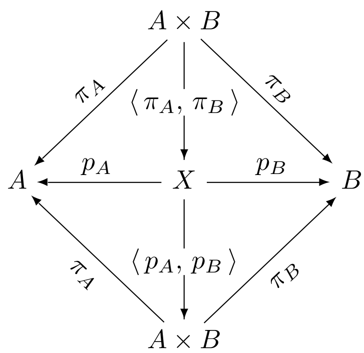
From what stated above it follows that there exists an arrow $\langle\, \pi_A,\, \pi_B \,\rangle$ such that

$$h = \langle\, \pi_A,\, \pi_B \,\rangle \equiv p_A \circ h = \pi_A \wedge p_B \circ h = \pi_B$$

From Definition 5.1 it follows also that there exists an arrow $\langle p_A, p_B \rangle$ such that

$$h = \langle p_A, p_B \rangle \equiv \pi_A \circ h = p_A \wedge \pi_B \circ h = p_B$$

We draw the arrows in the following diagram.



We prove that $\langle p_A, p_B \rangle \circ \langle \pi_A, \pi_B \rangle = \iota_{A\times B}$.

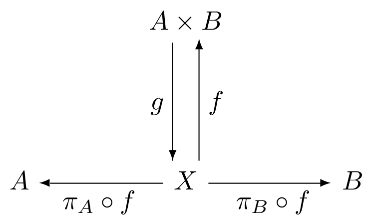
$$\begin{array}{ll}
& \langle p_A, p_B \rangle \circ \langle \pi_A, \pi_B \rangle = \iota_{A\times B} \\
\equiv & \{\text{First axiom } \circ\} \\
& \pi_A \circ (\langle p_A, p_B \rangle \circ \langle \pi_A, \pi_B \rangle) = \pi_A \circ \iota_{A\times B} \\
\equiv & \{\text{By unit axioms}\} \\
& \pi_A \circ (\langle p_A, p_B \rangle \circ \langle \pi_A, \pi_B \rangle) = \pi_A \\
\equiv & \{\text{By universal property}\} \\
& p_A \circ \langle \pi_A, \pi_B \rangle) = \pi_A \\
\equiv & \{\text{By universal property}\} \\
& \pi_A = \pi_A \\
\equiv & \{\text{By identity}\} \\
& \mathit{true}
\end{array}$$

Now we prove the second equality, $\langle \pi_A, \pi_B \rangle \circ \langle p_A, p_B \rangle = \iota_X$.

$$\begin{array}{ll}
& \langle \pi_A, \pi_B \rangle \circ \langle p_A, p_B \rangle = \iota_X \\
\equiv & \{\text{First axiom } \circ\} \\
& p_A \circ \langle \pi_A, \pi_B \rangle \circ \langle p_A, p_B \rangle = p_A \circ \iota_X \\
\equiv & \{\text{By unit axioms}\} \\
& p_A \circ \langle \pi_A, \pi_B \rangle \circ \langle p_A, p_B \rangle = p_A \\
\equiv & \{\text{By universal property}\} \\
& \pi_A \circ \langle p_A, p_B \rangle = p_A \\
\equiv & \{\text{By universal property}\} \\
& p_A = p_A \\
\equiv & \{\text{By identity}\} \\
& \mathit{true}
\end{array}$$

This concludes part (a); now we solve part (b) of the exercise.

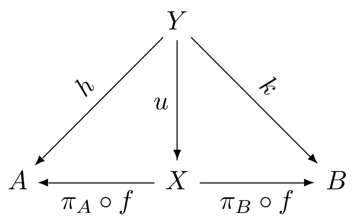
Assume an object X to exist, such that $X \cong A \times B$. By definition of isomorphism there exist two arrows $f : X \longrightarrow A \times B$ and $g : A \times B \longrightarrow X$, such that $f \circ g = \iota_{A \times B}$ and $g \circ f = \iota_X$. We have the diagram



To prove that the triple X, $\pi_A \circ f : X \longrightarrow A$, and $\pi_B \circ f : X \longrightarrow B$ is a product of A and B we have to show that for every object Y such that there exists two arrows $h : Y \longrightarrow A$ and $k : Y \longrightarrow B$, there *exists* an arrow $u : Y \longrightarrow X$ such that

$$l = u \equiv (\pi_A \circ f) \circ l = h \wedge (\pi_B \circ g) \circ l = k \tag{5}$$

In other words, we have to show that there exists a unique arrow k that makes the following diagram commute.



We show that there exists an arrow u such that

$$(\pi_A \circ f) \circ u = h \wedge (\pi_B \circ f) \circ u = k \tag{6}$$

Let $u = g \circ \langle h, k \rangle$, where the arrow $\langle h, k \rangle$ exists by definition of product and has the universal property

$$l = \langle h, k \rangle \equiv \pi_A \circ l = h \wedge \pi_B \circ l = k$$

We prove that u enjoys the property 6.

$$\begin{array}{ll}
 & (\pi_A \circ f) \circ u = h \wedge (\pi_B \circ f) \circ u = k \\
\equiv & \{\text{By Definition}\} \\
 & (\pi_A \circ f) \circ (g \circ \langle h, k \rangle) = h \wedge (\pi_B \circ f) \circ (g \circ \langle h, k \rangle) = k \\
\equiv & \{\text{By associativity}\} \\
 & \pi_A \circ (f \circ g) \circ \langle h, k \rangle = h \wedge \pi_B \circ (f \circ g) \circ \langle h, k \rangle = k \\
\equiv & \{\text{By Definition of isomorphism}\} \\
 & \pi_A \circ \iota_{A \times B} \circ \langle h, k \rangle = h \wedge \pi_B \circ \iota_{A \times B} \circ \langle h, k \rangle = k \\
\equiv & \{\text{By unit axioms}\} \\
 & \pi_A \circ \langle h, k \rangle = h \wedge \pi_B \circ \langle h, k \rangle = k \\
\equiv & \{\text{By UMP } \langle h, k \rangle\} \\
 & true \wedge true
\end{array}$$

Since (6) is true, we have shown that

$$l = u \Rightarrow (\pi_A \circ f) \circ l = h \wedge (\pi_B \circ f) \circ l = k$$

and we have to show the converse implication, which guarantees the uniqueness of u.

$$
\begin{array}{ll}
& (\pi_A \circ f) \circ l = h \wedge (\pi_B \circ f) \circ l = k \\
\equiv & \{\text{By (6)}\} \\
& (\pi_A \circ f) \circ l = (\pi_A \circ f) \circ u \wedge (\pi_B \circ f) \circ l = (\pi_B \circ f) \circ u \\
\equiv & \{\text{By Associativity}\} \\
& \pi_A \circ (f \circ l) = \pi_A \circ (f \circ u) \wedge \pi_B \circ (f \circ l) = \pi_B \circ (f \circ u) \\
\equiv & \{\text{By Lemma 5.9}\} \\
& f \circ l = f \circ u \wedge f \circ l = f \circ u \\
\equiv & \{\text{Composition preserves equality}\} \\
& g \circ (f \circ l) = g \circ (f \circ u) \wedge g \circ (f \circ l) = g \circ (f \circ u) \\
\equiv & \{\text{Associativity}\} \\
& (g \circ f) \circ l = (g \circ f) \circ u \wedge (g \circ f) \circ l = (g \circ f) \circ u \\
\equiv & \{\text{definition of isomorphism}\} \\
& \iota_X \circ l = \iota_X \circ u \wedge \iota_X \circ l = \iota_X \circ u \\
\equiv & \{\text{weakening}\} \\
& \iota_X \circ l = \iota_X \circ u \\
\equiv & \{\text{unit axioms}\} \\
& l = u
\end{array}
$$

□

Exercise 5.12.

[Associativity of products]
no. **2.8** (**13**) in **[Awo10]**

In any category with binary products, show directly that

$$A \times (B \times C) \cong (A \times B) \times C$$

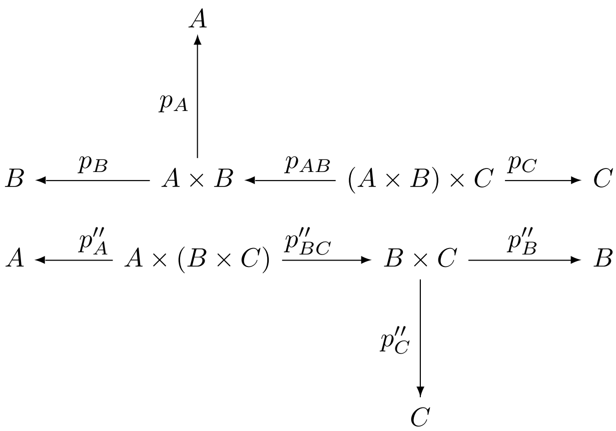
Proof. We have to show that there exists an isomorphism between the objects $A \times (B \times C)$ and $(A \times B) \times C$. This means that we have to exhibit two arrows

$$A \times (B \times C) \underset{g}{\overset{f}{\rightleftarrows}} (A \times B) \times C$$

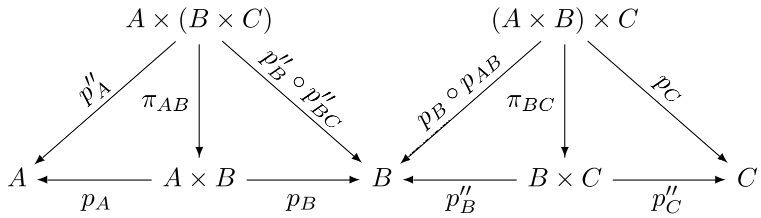
such that

$$f \circ g = \iota_{(A \times B) \times C}$$
$$g \circ f = \iota_{A \times (B \times C)}$$

We begin by showing some of the relations between the ternary and the binary products at issue. The following *projection* arrows exist



From the definition of product it follows that the arrows π_{BC} and π_{AB} exists in the diagram below

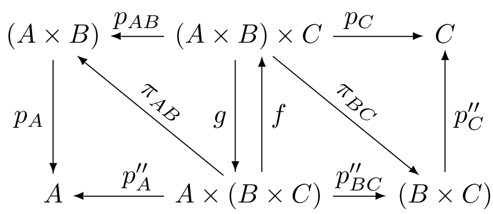


Moreover they enjoy the following properties

$$h = \pi_{BC} \quad \equiv \quad p''_B \circ h = p_B \circ p_{AB} \wedge p''_C \circ h = p_C \tag{7}$$

$$h = \pi_{AB} \quad \equiv \quad p_A \circ h = p''_A \wedge p_B \circ h = p''_B \circ p''_{BC} \tag{8}$$

Consider the following diagram



By definition of product, there exist two arrows f and g such that

$$h = f \quad \equiv \quad p_{AB} \circ h = \pi_{AB} \wedge p_C \circ h = p''_C \circ p''_{BC} \tag{9}$$

$$h = g \quad \equiv \quad p''_A \circ h = p_A \circ p_{AB} \wedge p''_{BC} \circ h = \pi_{BC} \tag{10}$$

Lemma 5.9 gives us a proof methods to show that $g \circ f = \iota_{A\times(B\times C)}$; namely, we are requires to prove the following equalities

$$\begin{aligned} p''_A \circ (g \circ f) &= p''_A \circ \iota_{A\times(B\times C)} \\ p''_{BC} \circ (g \circ f) &= p''_{BC} \circ \iota_{A\times(B\times C)} \end{aligned}$$

To prove the second equality we avail again of Lemma 5.9, and so we have to prove the equalities

$$\begin{aligned} P''_B \circ (P''_{BC} \circ (g \circ f)) &= p''_B \circ (P''_{BC} \circ \iota_{A\times(B\times C)}) \\ P''_C \circ (P''_{BC} \circ (g \circ f)) &= p''_C \circ (P''_{BC} \circ \iota_{A\times(B\times C)}) \end{aligned}$$

We give the proofs of the three equalities below.

$$\begin{array}{ll} & true \\ \equiv & \{\text{identity}\} \\ & p''_A = p''_A \\ \equiv & \{\text{definition product}\} \\ & p_A \circ \pi_{AB} = p''_A \\ \equiv & \{(9)\} \\ & p_A \circ (p_{AB} \circ f) = p''_A \\ \equiv & \{\text{associativity}\} \\ & (p_A \circ p_{AB}) \circ f = p''_A \\ \Rightarrow & \{(10)\} \\ & (p''_A \circ g) \circ f = p''_A \\ \equiv & \{\text{unit axioms}\} \\ & (p''_A \circ g) \circ f = p''_A \circ \iota_{A\times(B\times C)} \\ \equiv & \{\text{associativity}\} \\ & p''_A \circ (g \circ f) = p''_A \circ \iota_{A\times(B\times C)} \end{array}$$

$$\begin{array}{ll} & true \\ \equiv & \{\text{identity}\} \\ & p''_B = p''_B \\ \equiv & \{\text{Composition preserves equalities}\} \\ & p''_B \circ p''_{BC} = p''_B \circ p''_{BC} \\ \Rightarrow & \{\text{UMP } \pi_{AB};\ (8)\} \\ & p''_B \circ p''_{BC} = p_B \circ \pi_{AB} \\ \Rightarrow & \{\text{UMP } f;\ (9)\} \\ & p''_B \circ p''_{BC} = p_B \circ (p_{AB} \circ f) \\ \equiv & \{\text{Associativity}\} \\ & p''_B \circ p''_{BC} = (p_B \circ p_{AB}) \circ f \\ \Rightarrow & \{\text{UMP } \pi_{BC};\ (7)\} \\ & p''_B \circ p''_{BC} = (p''_B \circ \pi_{BC}) \circ f \\ \Rightarrow & \{\text{UMP } g;\ (10)\} \\ & p''_B \circ p''_{BC} = (p''_B \circ (p''_{BC} \circ g)) \circ f \\ \equiv & \{\text{Associativity}\} \\ & p''_B \circ p''_{BC} \circ \iota_{A\times(B\times C)} = p''_B \circ p''_{BC} \circ (g \circ f) \end{array}$$

$$
\begin{array}{ll}
 & true \\
\equiv & \{\text{identity}\} \\
 & p''_C = p''_C \\
\equiv & \{\text{Composition preserves equality}\} \\
 & p''_C \circ p''_{BC} = p''_C \circ p''_{BC} \\
\equiv & \{\text{Composition preserves equality}\} \\
 & p''_C \circ p''_{BC} \circ g = p''_C \circ p''_{BC} \circ g \\
\Rightarrow & \{\text{UMP } g; (10)\} \\
 & p''_C \circ \pi_{BC} = p''_C \circ p''_{BC} \circ g \\
\equiv & \{\text{Composition preserves equality}\} \\
 & (p''_C \circ \pi_{BC}) \circ f = (p''_C \circ p''_{BC} \circ g) \circ f \\
\Rightarrow & \{\text{UMP } \pi_{BC}; (7)\} \\
 & p_C \circ f = p''_C \circ p''_{BC} \circ g \circ f \\
\Rightarrow & \{\text{UMP } f; (9)\} \\
 & p''_C \circ p''_{BC} = p''_C \circ p''_{BC} \circ g \circ f \\
\equiv & \{\text{Unit axiom}\} \\
 & (p''_C \circ p''_{BC}) \circ \iota_{A\times(B\times C)} = p''_C \circ p''_{BC} \circ (g \circ f)
\end{array}
$$

We have proven the three equalities that, via Lemma 5.9 let us state that $g \circ f = \iota_{A\times(B\times C)}$.

We do not give the proof that $f \circ g = \iota_{(A\times B)\times C}$, because it is analogous to the argument we have used. □

We prove other properties of products.

Lemma 5.13. Let **C** be a category with finite products. Then we have that

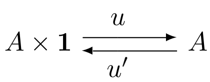
(i) the terminal object is the unit: $A \times \mathbf{1} \cong A$

(ii) the identity of a product is the product map of identities of the components: $\iota_{A\times B} = \iota_A \times \iota_B$

(iii) if $A \cong B$ and $B \cong D$ then $A \times C \cong B \times D$

Proof. We organise the proof following the enumeration in the statement of the lemma.

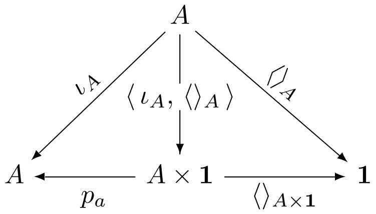
(i) We have to exhibit two arrows



such that

$$u \circ u' = \iota_A \tag{11}$$
$$u' \circ u = \iota_{A\times\mathbf{1}} \tag{12}$$

Consider the diagram



and let $u = p_A$ and $u' = \langle\, \iota_A,\, \langle\rangle_A \,\rangle$. Note that the objects and the arrows in the diagram above exists because we are under the hypothesis that **C** has finite products, so also the empty one, which is the terminal object **1**. By definition of product we have the following property of $\langle\, \iota_A,\, \langle\rangle_A \,\rangle$,

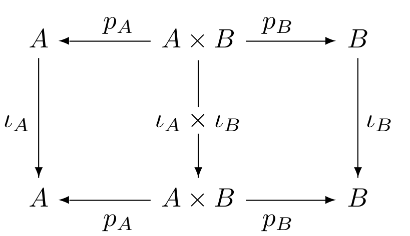
$$h = \langle\, \iota_A,\, \langle\rangle_A \,\rangle \equiv p_A \circ h = \iota_A \wedge \langle\rangle_A \circ h = \langle\rangle_A$$

The equality (11) follows in a straightforward way: $p_A \circ \langle\, \iota_A,\, \langle\rangle_A \,\rangle = \iota_A$.

We prove (12).

$$\begin{array}{ll}
 & \langle\, \iota_A,\, \langle\rangle_A \,\rangle \circ p_A = \iota_{A\times\mathbf{1}} \\
\equiv & \{\text{Fusion law}\} \\
 & \langle\, \iota_A \circ p_A,\, \langle\rangle_A \circ p_A \,\rangle = \iota_{A\times\mathbf{1}} \\
\equiv & \{\text{Unit axioms}\} \\
 & \langle\, p_A,\, \langle\rangle_A \circ p_A \,\rangle = \iota_{A\times\mathbf{1}} \\
\equiv & \{\text{Universal property final objects}\} \\
 & \langle\, p_A,\, \langle\rangle_{A\times\mathbf{1}} \,\rangle = \iota_{A\times\mathbf{1}} \\
\equiv & \{\text{Universal property products}\} \\
 & p_A \circ \iota_{A\times\mathbf{1}} = p_A \\
 & \wedge \\
 & \langle\rangle_{A\times\mathbf{1}} \circ \iota_{A\times\mathbf{1}} = \langle\rangle_{A\times\mathbf{1}} \\
\equiv & \{\text{Unit axioms}\} \\
 & p_A = p_A \\
 & \wedge \\
 & \langle\rangle_{A\times\mathbf{1}} = \langle\rangle_{A\times\mathbf{1}} \\
\equiv & \{\text{Identity}\} \\
 & \mathit{true} \wedge \mathit{true}
\end{array}$$

(ii) We have to show that $\iota_{A\times B} = \iota_A \times \iota_B$. First of all note that the symbol $\times$ has more than one meaning; in particular $\iota_A \times \iota_B$ stands for the unique arrow $\langle\, \iota_A \circ p_A,\, \iota_B \circ p_B \,\rangle$ in the diagram

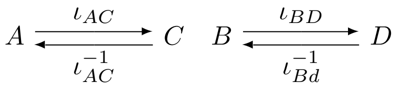


such that the diagram commutes. In reason of the uniqueness of $\iota_A \times \iota_B$, in order to prove that $\iota_{A\times B} = \iota_A \times \iota_B$ it suffices to show that also $\iota_{A\times B}$ makes the diagram commute. This follows form the identity axiom; the proof that $p_A \circ \iota_{A\times B} = \iota_A \circ p_A$ amounts in two steps

$$p_A \circ \iota_{A\times B} = p_A = \iota_A \circ p_A$$

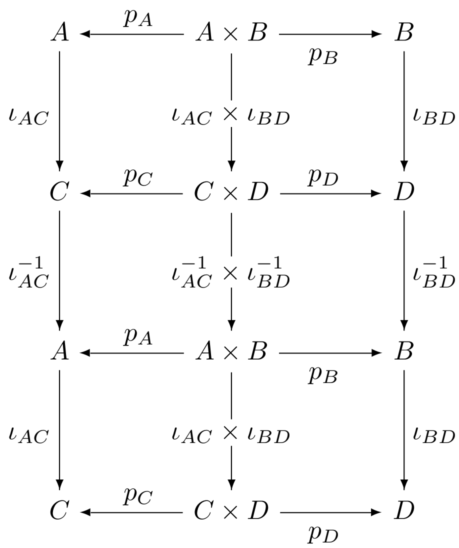
and so does the proof of $p_B \circ \iota_{A\times B} = \iota_B \circ p_B$.

(iii) The proof of the last point is made by two parts. First we prove that if $A \cong C$ and $B \cong D$ then $A \times B \cong C \times D$. Let



be the isomorphisms respectively between A and C, and B and D.

Consider the diagram



We prove that $\iota_{AC} \times \iota_{BD} \circ \iota_{AC}^{-1} \times \iota_{BD}^{-1} = \iota_{C \times D}$. Remember that thanks to Exercise 5.8, $(f \times h) \circ \langle g, k \rangle = \langle f \circ g, h \circ k \rangle$.

$$
\begin{aligned}
\iota_{AC} \times \iota_{BD} \circ \iota_{AC}^{-1} \times \iota_{BD}^{-1} =& \iota_{AC} \times \iota_{BD} \circ \langle \iota_{AC}^{-1} \circ p_C, \iota_{BD}^{-1} \circ P_D \rangle \\
=& \langle \iota_{AC} \circ \iota_{AC}^{-1} \circ p_C, \iota_{BD} \circ \iota_{BD} \circ p_D \rangle \\
=& \langle \iota_C \circ p_C, \iota_D \circ p_D \rangle \\
=& \iota_C \times \iota_D \\
=& \iota_{C \times D}
\end{aligned}
$$

The last passage is justified by the previous point of the lemma. The proof that $\iota_{AC}^{-1} \times \iota_{BD}^{-1} \circ \iota_{AC} \times \iota_{BD} = \iota_{A \times B}$ is analogous.

□

We give a result will ease the solution of the next exercise.

Proposition 5.14. Let **C** be a category with empty product and binary products. Then **C** has finite products.

Proof. A finite product of objects can be denoted

$$((O_1 \times O_2) \times \ldots \times O_{n-1}) \times O_n$$

for some $n \in \mathbb{N} \cup \{0\}$. To prove the proposition we have therefore to show that for each n the product above is defined. Let P_n a product of n objects; we reason by induction on n. If $n = 0$ then the product P_n is the empty one, and by hypothesis it is defined.

If $n > 0$ then

$$P_n = ((O_1 \times O_2) \times \ldots \times O_{n-1}) \times O_n$$

let $P_{n-1} = ((O_1 \times O_2) \times \ldots \times O_{n-1})$. The proof that P_n is defined amounts in showing that $P_{n-1} \times O_n$ is defined. By inductive hypothesis the product P_{n-1} is defined; moreover, O_n is defined, for it is an object. The hypothesis of **C** having binary products implies that $P_{n-1} \times O_n$ is defined, because the objects it involves are defined. We have thus shown that the finite product P_n is defined. Since we have used no particular assumptions on P_n we have indeed proven that all finite products are defined. □

Exercise 5.15.
no. **2.6.7** (**1**) in **[Cro94]**

Show that a category **C** has finite products just in case it has binary products and a terminal object.

Proof. Suppose **C** has finite products; then it must have binary products and the terminal object, because they are (isomorphic to) products of a finite number of objects. If **C** has binary products and terminal object then we use Proposition 5.14. □

Exercise 5.16.
no. **3.5** (**6**) in **[Awo10]**

Verify that the category of monoids has all finite products, then do the same for abelian groups.

Proof. We limit our solution to monoids.

Now we have to prove that the category **Monoids** has finite products. Thanks to Proposition 5.14 all we have to do is to check that the category **Monoids** has empty product and binary products.

Lemma 1.4 ensures that the category **Monoids** has terminal object **1**; that is the empty product.

As to the existence of binary products, Lemma 1.5 ensures that the object $\langle M_1 \times M_2, *, (u_1, u_2) \rangle$ and the arrows $fst : \langle M_1 \times M_2, * \rangle \longrightarrow \langle M_1, \cdot \rangle$ and $snd : \langle M_1 \times M_2, * \rangle \longrightarrow \langle M_1, \cdot \rangle$ exist in **Monoids**. We say that the product of the monoids $\langle M_1, \cdot, u_1 \rangle$ and $\langle M_2, +, u_2 \rangle$ is the object $\langle M_1 \times M_2, *, u' \rangle$ with the projection arrows fst and snd.

The fact that we have *called* the tuple $(\langle M_1 \times M_2, *, (u_1, u_2) \rangle, fst, snd)$ a product does not mean that this tuple *is* a product; a proof is in order.

We have to show that if for some monoid $\langle N, \circ, u_N \rangle$ in the category **Monoids** we have

$$\langle M_1, \cdot, u_1 \rangle \xleftarrow{f} \langle N, \circ, u_N \rangle \xrightarrow{g} \langle M_2, \times, u_2 \rangle$$

then there exists a unique arrow $\langle f, g \rangle : \langle N, \circ \rangle \longrightarrow \langle M_1 \times M_2, * \rangle$ such that $f = fst \circ \langle f, g \rangle$ and $g = snd \circ \langle f, g \rangle$.

- **Existence**
  Consider the function $\langle f, g \rangle(x) = (f(x), g(x))$ for every $x \in M_1 \times M_2$; Lemma 1.6 ensures that the arrow

$$\langle f, g \rangle : \langle N, \circ, u_N \rangle \longrightarrow \langle M_1 \times M_2, *, (u_1, u_2) \rangle$$

  exists in **Monoids**.

  We prove that $f = fst \circ \langle f, g \rangle$ and $g = snd \circ \langle f, g \rangle$; if $x \in N$ then

$$fst(\langle f, g \rangle(x)) = fst((f(x), g(x))) = f(x)$$
$$snd(\langle f, g \rangle(x)) = snd((f(x), g(x))) = g(x)$$

  We have proven

$$h = \langle f, d \rangle \Rightarrow fst \circ h = f \wedge snd \circ h = g \tag{13}$$

- **Uniqueness**
  Suppose that for some arrow $u' : \langle N, \circ, u_N \rangle \longrightarrow \langle M_1 \times M_2, *, (u_1, u_2) \rangle$ we have $f = fst \circ u'$ and $g = snd \circ u'$. We prove that $u = u'$.

$$\begin{array}{ll} & f = fst \circ u' \wedge g = snd \circ u' \\ \Rightarrow & \{13\} \\ & fst \circ u' = fst \circ \langle f, g \rangle \wedge snd \circ u' = snd \circ \langle f, g \rangle \\ \equiv & \{\text{definition of } fst \text{ and } snd\} \\ & u' = \langle f, g \rangle \end{array}$$

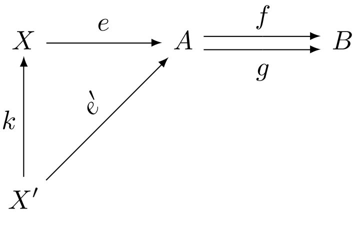
□

6 Equalisers

Definition 6.1. [Equalizer, [Pie91, Cro94]]
An arrow $e : X \longrightarrow A$ is an *equalizer* of a pair of arrows $f : A \longrightarrow B$ and $g : A \longrightarrow B$ if

1) $f \circ e = g \circ e$

2) whenever $e' : X' \longrightarrow A$ satisfies $f \circ e' = g \circ e'$, then there is a unique arrow $k : X' \longrightarrow X$ such that $e \circ k = e'$:



□

Exercise 6.2.
no. **1.7.4 (1)** in **[Pie91]**

Show that in a poset considered as a category, the only equalizers are the identity arrows.

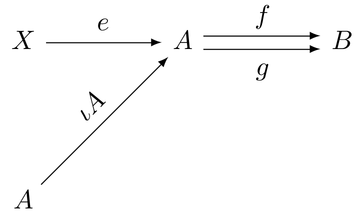
Proof. Let $\langle P, \leq_P \rangle$ be a poset (see Definition 1.2) and let $\leq_{\mathbf{P}}$ the category that arises form it. We have to prove two things,

(a) if e is an equalizer of a pair of arrows f and g in $\leq_{\mathbf{P}}$, then e is an identity arrow.

(b) for every object A the identity arrow ι_A is an equalizer of every two arrows $f : A \longrightarrow B$, $g : A \longrightarrow B$

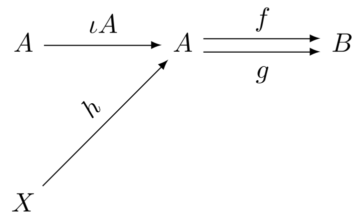
We prove in order (a) and (b).

Consider the following diagram



where e is an equalizer of f and g. By Proposition 2.5 ensures that $f = g$, and thus $f \circ \iota A = g \circ \iota A$. We know by definition of equalizer that there exists $k : A \longrightarrow X$. The arrow e is the pair (X, A), and the arrow k is the pair (A, X); since $\leq_P$ is a poset it follows that $A = X$, and therefore Proposition 2.5 ensures that $e = \iota A$.

Now we prove point (b). Let A be an object of $\leq_{\mathbf{P}}$ and ιA the identity arrow on it; moreover, let $f : A \longrightarrow B$ and $g : A \longrightarrow B$ two arrows in $\leq_{\mathbf{P}}$ such that $f \circ \iota A = g \circ \iota A$. Consider now the following diagram



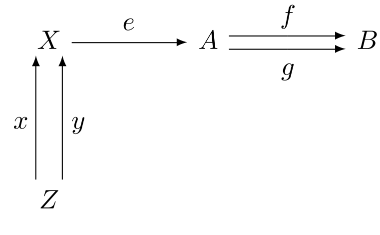
where $f \circ h = g \circ h$. We have to prove that there exists a unique arrow $k : X \longrightarrow A$ such that $k \circ \iota A = k$. Note that thanks to Proposition 2.5 it is enough to prove that there exists such a k, and thanks to Proposition 2.6 it is enough to show the exists of a $k : X \longrightarrow A$. As k we choose h, that is $k = h$, for h exists.

In the proof of point (b) we have not used the anti-symmetry, thus the property holds for the category given by any preorder. □

Exercise 6.3.
no. **1.7.4 (2)** in **[Pie91]**

Show that every equalizer is monic.

Proof. Consider the diagram below



With $e : X \longrightarrow A$ an equalizer. We have to prove that if $e \circ x = e \circ y$ then $x = y$.

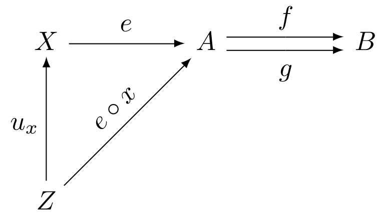
The definition of equalizer implies that $f \circ e = g \circ e$; we use the first axiom of categories to derive

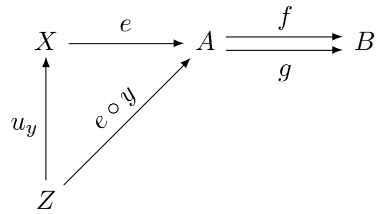
$$\begin{gathered}(f \circ e) \circ y = (g \circ e) \circ y \\ \wedge \\ (f \circ e) \circ x = (g \circ e) \circ x\end{gathered}$$

and associativity ensures that

$$\begin{gathered}f \circ (e \circ y) = g \circ (e \circ y) \\ \wedge \\ f \circ (e \circ x) = g \circ (e \circ x)\end{gathered}$$

Since e is an equalizer, the equalities above imply that there **exist** two **unique** arrows u_x, u_y such that the diagrams below commute





We have that

$$\begin{aligned}e \circ u_x &= e \circ x \\ e \circ u_y &= e \circ y\end{aligned}$$

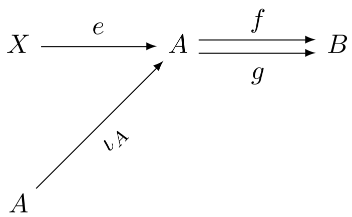
The uniqueness of u_x ensures that $u_x = x$. The hypothesis $e \circ x = e \circ y$ implies that $e \circ u_x = e \circ y$, so the uniqueness of u_x ensures that $u_x = y$; we have thus $x = u_x = y$. □

Exercise 6.4.

no. **1.7.4** (**3**) in **[Pie91]**

Show that every epic equalizer is an isomorphism.

Proof. Consider the diagram below



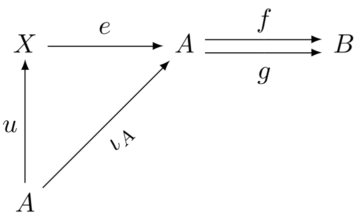
where, by hypothesis, the arrow $e : X \longrightarrow A$ is an equalizer and it is epic. We have to prove that e is an isomorphism; this amounts in proving that there *exists* an arrow $u : A \longrightarrow X$ such that

$$e \circ u = \iota_A \qquad u \circ e = \iota_X$$

We begin by assessing which arrows do exist and what properties they enjoy. Since e is an equalizer we know that $f \circ e = g \circ e$; now we use the fact that e is epic to show that $f = g$. From the last equality and the first axiom it follows the equality $f \circ \iota_A = g \circ \iota_A$. Now the hypothesis that e is an equalizer ensures that there exists a unique $u : A \longrightarrow X$ such that

$$\iota_A = e \circ u \tag{14}$$

We have the diagram below



Half of the proof is complete, as we have shown that there exists an arrow u such that $e \circ u = \iota_A$. What we have left to do, is to show that $u \circ e = \iota_X$.

$$\begin{array}{ll}
 & \mathit{true} \\
\equiv & \{\text{Identity}\} \\
 & e = e \\
\equiv & \{\text{Unit axiom}\} \\
 & \iota_A \circ e = e \\
\equiv & \{\text{(14) above}\} \\
 & (e \circ u) \circ e = e \\
\equiv & \{\text{Associativity}\} \\
 & e \circ (u \circ e) = e \\
\equiv & \{\text{Identity axiom}\} \\
 & e \circ (u \circ e) = e \circ \iota_X \\
\Rightarrow & \{\text{equalizers are monic}\} \\
 & u \circ e = \iota_X
\end{array}$$

□

The next two lemmas are not in the referenced books. I need them in order to prove results mentioned in all the books, but proven in ways which I do not understand.

Lemma 6.5. In the category **Sets**, let A and B be objects with arrows $f : A \longrightarrow B$ and $g : A \longrightarrow B$. Let S_{fg} be the set

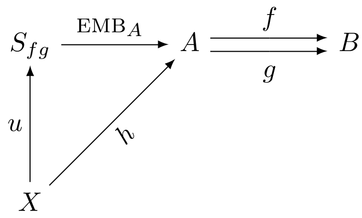
$$S_{fg} = \{\, a \in A \mid f(a) = g(a) \,\}$$

and $\textsc{emb}_A : S_{fg} \longrightarrow A$ the arrow defined by

$$\textsc{emb}_A(s) = s$$

The arrow $\textsc{emb}_A$ is an equalizer for the arrows f and g.

Proof. We have to prove that for each object X, if there exists an arrow $h : X \longrightarrow A$ such that $f \circ h = g \circ h$, then there exists a unique arrow $u : X \longrightarrow S_{fg}$ such that $h = \textsc{emb}_A \circ u$. Suppose that we have objects and arrows as in the diagram



and that $f \circ h = g \circ h$.

We divide the argument in two parts; firstly, we show that an arrow u exists; secondly, we prove that it is unique.

- **Existence**
  Let $u : X \longrightarrow S_{fg}$ be the arrow defined by

  $$u(x) = h(x)$$

  for every $x \in X$. We have to show that the equality $h = \textsc{emb}_A \circ u$.

  $$\begin{array}{ll} & h = \textsc{emb}_A \circ u \\ \equiv & \{\text{definition of arrow in } \mathbf{Sets}\} \\ & \forall x \in X.h(x) = (\textsc{emb}_A \circ u)(x) \\ \equiv & \{\text{definition of function composition}\} \\ & \forall x \in X.h(x) = \textsc{emb}_A(u(x)) \\ \equiv & \{\text{definition of } \textsc{emb}_A\} \\ & \forall x \in X.h(x) = u(x) \\ \equiv & \{\text{definition of } u\} \\ & \mathit{true} \end{array}$$

  We have proven the existence of a suitable arrow u, now we move on and we show that it is unique.

- **Uniqueness** Let $u' : X \longrightarrow S_{fg}$ be an arrow such that $h = \textsc{emb}_A \circ u'$; we show that $u = u'$.

  $$\begin{array}{ll} & h = \textsc{emb}_A \circ u' \\ \equiv & \{\text{construction}\} \\ & \textsc{emb}_A \circ u = \textsc{emb}_A \circ u' \\ \equiv & \{\textsc{emb}_A \text{ is monic}\} u = u' \end{array}$$

  The fact that the arrow $\textsc{emb}_A$ is monic derives from the fact that $\textsc{emb}_A$ acts on the elements of the set S_{fg} as the identity.

□

Lemma 6.6. Let $e : E \longrightarrow A$ and $e' : E' \longrightarrow A$ two equalizers of the arrows in the diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

The objects E and E' are isomorphic.

Proof. A pair of arrows $u : E \longrightarrow E'$ and $u' : E' \longrightarrow E$ has to be shown, which satisfy the following equalities

$$u \circ u' = \iota_{E'} \quad u' \circ u = \iota_E$$

We have to exhibit such u and u'.

The definition of equalizer implies that there exist two unique arrows u and u' such that

$$e' \circ u = e, \quad e \circ u' = e'$$

The proof that $u' \circ u = \iota_E$ is as follows

$$\begin{array}{ll} & e' \circ u = e \\ \equiv & \{\text{equality above}\} \\ & (e \circ u') \circ u = e \\ \equiv & \{\text{associativity}\} \\ & e \circ (u' \circ u) = e \\ \equiv & \{\text{unit axioms}\} \\ & e \circ (u' \circ u) = e \circ \iota_E \\ \equiv & \{\text{equalizers are monic}\} \\ & u' \circ u = \iota_E \end{array}$$

And here is the proof that $u \circ u' = \iota_{E'}$

$$\begin{array}{ll} & e \circ u' = e' \\ \equiv & \{\text{equality above}\} \\ & (e' \circ u) \circ u' = e' \\ \equiv & \{\text{associativity}\} \\ & e' \circ (u \circ u') = e' \\ \equiv & \{\text{unit axioms}\} \\ & e' \circ (u \circ u') = e' \circ \iota_{E'} \\ \equiv & \{\text{equalizers are monic}\} \\ & u \circ u' = \iota_{E'} \end{array}$$

□

In [Pie91] and [Awo10] the authors remarks that subsets represents equalizers in the category **Sets**. In [Pie91], though, there is no proof; moreover I cannot understand the proof give in [Awo10]. I resolved myself to prove the following.

Proposition 6.7. Let us reason in the category **Sets**, and let S, A and B be sets.

1. If $S \subseteq A$ and $\text{EMB}_A : S \longrightarrow A$ is the total function $\text{EMB}_A(s) = s$, then EMB_A is an equalizer; note that the arrow name EMB_A is a mnemonic for "embed in A".
2. If $e : S \longrightarrow A$ is the equalizer of two arrows $f : A \longrightarrow B$ and $g : A \longrightarrow B$, then S is isomorphic to the set $S_{fg} = \{\, a \in A \mid f(a) = g(a) \,\}$.

Proof. The proof is divided in two parts, which mirror the structure of the proposition.

1. To begin with, we show that there exists a unique arrow $\top : \mathbf{1} \longrightarrow \mathbf{2}$ which represents the truth value TRUE.

   Consider the set $\mathbf{1}$ and the arrows from it to the set of truth values $\{$ TRUE, FALSE $\}$. Since the last set is isomorphic to the set $\mathbf{2}$, we shall denote it with the symbol $\mathbf{2}$. There are two arrows form $\mathbf{1}$ to $\mathbf{2}$, one of which is $\top = (T, \mathbf{1}, \mathbf{2})$, where T is the constant function $T(*) = \text{TRUE}$. We know also that there exists a unique arrow $\langle\rangle_A : A \longrightarrow \mathbf{1}$, because each element of A can be mapped only to the unique element of $\mathbf{1}$. It follows that we have the arrow $T \circ \langle\rangle_A : A \longrightarrow \mathbf{2}$

   Now, let us focus on the characteristic function of the set S; we denote this function χ_S and define it as

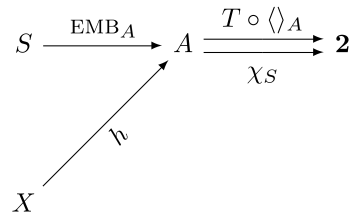
$$\chi_S(s) = \begin{cases} \text{TRUE} & \text{if } s \in S \\ \text{FALSE} & \text{otherwise} \end{cases}$$

   We abuse the notation and denote with χ_S also the arrow $(\chi_s, S, \mathbf{2})$. We shall prove that the arrow EMB_A is an equalizer for the arrows $\top \circ \langle\rangle_A$ and χ_S. First we are required to show that EMB_A indeed equalizes χ_s and $\top \circ \langle\rangle_A$, that is $\chi_s \circ \text{EMB}_A = (\top \circ \langle\rangle_A) \circ \text{EMB}_A$.

$$\begin{array}{ll} & \chi_s \circ \text{EMB}_A = (\top \circ \langle\rangle_A) \circ \text{EMB}_A \\ \equiv & \{\text{Meaning of arrow equality in } \textbf{Sets}\} \\ & \forall s \in S.(\chi_s \circ \text{EMB}_A)(s) = ((T \circ \langle\rangle_A) \circ \text{EMB}_A)(s) \\ \equiv & \{T \text{ constant}\} \\ & \forall s \in S.(\chi_s \circ \text{EMB}_A)(s) = \text{TRUE} \\ \equiv & \{\text{arrow composition}\} \\ & \forall s \in S.\chi_s(\text{EMB}_A(s)) = \text{TRUE} \\ \equiv & \{\text{definition of EMB}_A\} \\ & \forall s \in S.\chi_s(s) = \text{TRUE} \\ \equiv & \{\text{because } s \in S\} \\ & \forall s \in S.\text{TRUE} = \text{TRUE} \\ \equiv & \{s \text{ does not appear free in TRUE}\} \\ & \text{TRUE} = \text{TRUE} \\ \equiv & \{\text{Identity}\} \\ & \textit{true} \end{array}$$

   Suppose that there exists an object X and an arrow $h : X \longrightarrow A$ such that $\top \circ h = \chi_S \circ h$. We have to show that there exists a unique arrow $u : X \longrightarrow S$ such that $h = \text{EMB}_A \circ u$.

   Before unfolding the proof, we drawn in a diagram what we have discussed so far



- **Existence** We have to choose a suitable u.

  The left-hand side of the assumption $\top \circ h = \chi_S \circ h$ is constant, because so is $\top$, thus we have the equality $\top = \chi_S \circ h$. In functional terms, $\chi_S(h(x)) = \text{TRUE}$ for every $x \in X$. This means that if $x \in X$ then $h(x) \in S$.

  Let $u : X \longrightarrow S$ be the arrow given by the total function defined as $u(x) = h(x)$. We prove that u satisfies the required property, that is $h = \text{EMB}_A \circ u$. Indeed, if $x \in X$, then $\text{EMB}_A(u(x)) = \text{EMB}_A(h(x)) = h(x)$.

- **Uniqueness** Suppose that there was an arrow $u' : X \longrightarrow S$ such that $h = \text{EMB}_A \circ u'$.

$$
\begin{aligned}
& h = \text{EMB}_A \circ u' \\
\equiv\ & \{\text{Because } h = \text{EMB}_A \circ u\} \\
& \text{EMB}_A \circ u = \text{EMB}_A \circ u' \\
\equiv\ & \{\text{EMB}_A \text{ is monic}\} \\
& u = u'
\end{aligned}
$$

2. This point follows from Lemma (6.5) and (6.6).

□

7 Exponentials

Definition 7.1. [UMP exponentials]

$$h = \text{CURRY}(f) \equiv f = \text{EV}_{AB} \circ (h \times \iota_A)$$

□

Exercise 7.2.

Show that for alla finite sets M and N,

$$|N^M| = |N|^{|M|}$$

where $|K|$ is the number of elements in the set K, while N^M is the exponential in the category of sets (the set of all functions $f : M \longrightarrow N$), and n^m is the usual exponentiation operation of arithmetic.

Proof. The simplest argument is combinatotial and, at least on the surface, has nothing to do with categories.

How many elements are in the set N^M? Each function f in N^M has to be total, hence it has to map $|M|$ elements to some element of N. It follows that for each element of M there are $|N|$ choices. The number of different total function from M to N must therefore be

$$\underbrace{|N| \cdot |N| \cdot \ldots |N|}_{|M|\ times}$$

which is exactly $|N|^{|M|}$. □

Exercise 7.3.
no. **1.8.7** (**4**) in [**Pie91**]

Show that $\textsc{curry}(\textsc{ev}_{AB}) = \iota_{B^A}$.

Proof. In view of the UMP of exponentials, the proof is straightforward:

$$
\begin{aligned}
& \textsc{curry}(\textsc{ev}_{AB}) = \iota_{B^A} \\
\equiv\ & \{\text{UMP of exponentials}\} \\
& \textsc{ev}_{AB} = \textsc{ev}_{AB} \circ (\iota_{B^A} \times \iota_A) \\
\equiv\ & \{\text{Lemma 5.13}\} \\
& \textsc{ev}_{AB} = \textsc{ev}_{AB} \circ (\iota_{B^A \times A}) \\
\equiv\ & \{\text{Identity axiom}\} \\
& \textsc{ev}_{AB} = \textsc{ev}_{AB} \\
\equiv\ & \{\text{Identity}\} \\
& \mathit{true}
\end{aligned}
$$

□

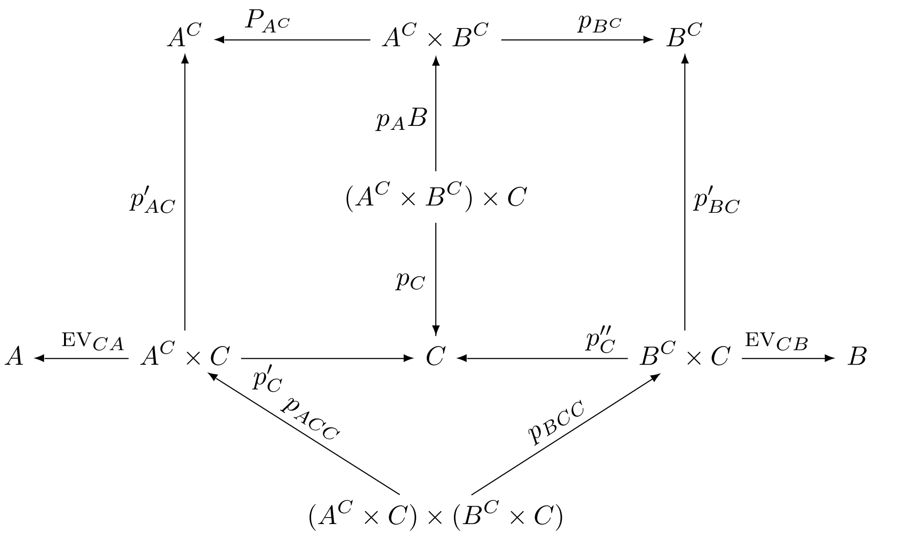
Proposition 7.4. In a CCC the object $A^C \times B^C$ is an exponential of $A \times B$ with respect to the object C.

Proof. We have to show that there exist an arrow

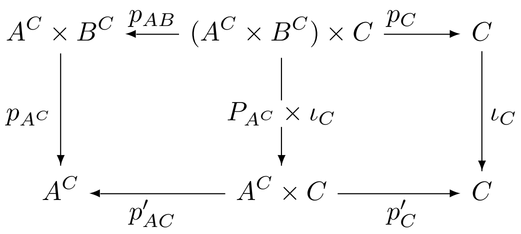
$$\textsc{ev}_{(AB)C} : A^C \times B^C \times C \longrightarrow A \times B$$

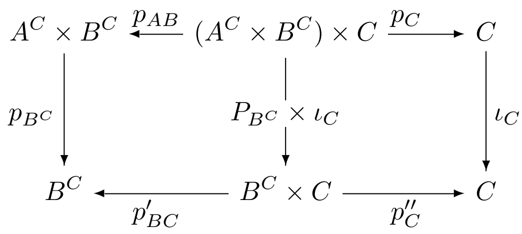
which satisfies the UMP of the exponential objects.

As we are in a category with products, we have the diagram

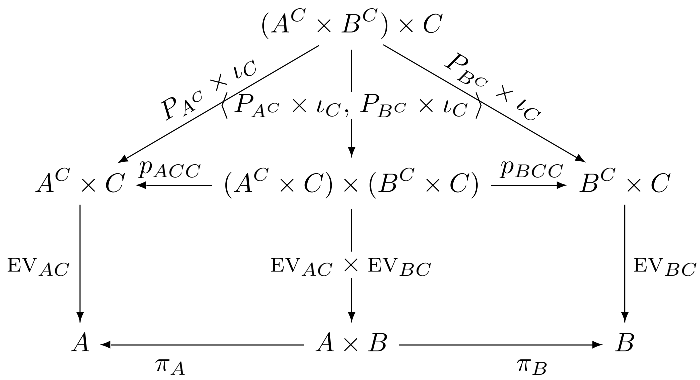


and so there exist two unique arrows $P_{A^C} \times \iota_C$ and $P_{B^C} \times \iota_C$ which makes the diagrams below commute





And so we have the diagram



with the arrow

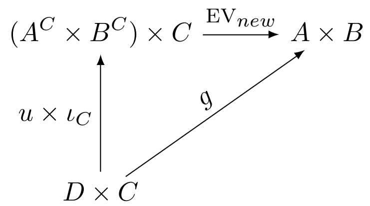
$$\begin{aligned}\text{EV}_{new} &= (\text{EV}_{AC} \times \text{EV}_{BC}) \circ \langle\, P_{A^C} \times \iota_C,\, P_{B^C} \times \iota_C \,\rangle \\ &= \langle\, \text{EV}_{AC} \circ (P_{A^C} \times \iota_C),\, \text{EV}_{BC} \circ (P_{B^C} \times \iota_C) \,\rangle\end{aligned}$$

which is the unique that makes the diagram commute; its UMP is

$$h = \text{EV}_{new} \equiv \pi_A \circ h = \text{EV}_{AC} \circ (P_{A^C} \times \iota_C) \wedge \pi_B \circ h = \text{EV}_{BC} \circ (P_{B^C} \times \iota_C)$$

Now we have to show that the object $A^C \times B^C$ with the arrow EV_{new} enjoys the UMP of the exponential of $A \times B$ by C.

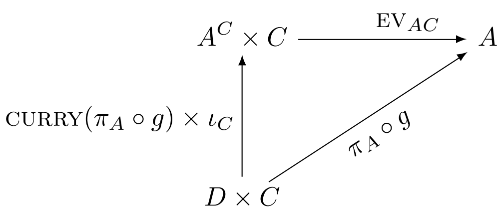
Suppose that there exist an object D and an arrow $g : D \times C \longrightarrow A \times B$; we have to prove that there exists a unique arrow $u : D \longrightarrow A^C \times B^C$ which makes the diagram below commute

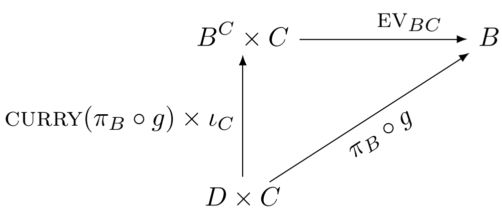


Algebraically

$$\text{EV}_{new} \circ (u \times \iota_C) = g$$

First we define u, and then we prove that it is unique. To begin with, we know that A^C and B^C are exponentials, and that there exist the arrows $\pi_A : A \times B \longrightarrow A$, and $\pi_B : A \times B \longrightarrow B$. It follows that there exists two unique arrows $\text{CURRY}(\pi_A \circ g)$ and $\text{CURRY}(\pi_B \circ g)$ which make the diagrams below commute.





and we have the UMP's

$$\begin{array}{rl} h = \text{CURRY}(\pi_A \circ g) \equiv & \pi_A \circ g = \text{EV}_{AC} \circ (h \times \iota_C) \\ h = \text{CURRY}(\pi_B \circ g) \equiv & \pi_B \circ g = \text{EV}_{BC} \circ (h \times \iota_C) \end{array}$$

We have proven that there exists the following arrows

$$A^C \xleftarrow{\text{CURRY}(\pi_A \circ g)} D \xrightarrow{\text{CURRY}(\pi_B \circ g)} B^C$$

and, as we are in a category with products, it follows that there is an arrow $\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle$ such that

$$h = \langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \equiv p_{A^C} \circ h = \text{CURRY}(\pi_A \circ g) \wedge p_{B^C} \circ h = \text{CURRY}(\pi_B \circ g)$$

The arrow $\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle$ is the one we are after. We prove that $\text{EV}_{new} \circ (\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \times \iota_C) = g$.

$$\begin{array}{ll}
 & \text{EV}_{new} \circ (\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \times \iota_C) = g \\
\equiv & \{\text{Composition with projections and Lemma 5.9}\} \\
 & \pi_A \circ \text{EV}_{new} \circ (\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \times \iota_C) = \pi_A \circ g \\
 & \wedge \\
 & \pi_B \circ \text{EV}_{new} \circ (\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \times \iota_C) = \pi_B \circ g \\
\equiv & \{\text{UMP } \text{EV}_{new}\} \\
 & \text{EV}_{AC} \circ (p_{A^C} \times \iota_C) \circ (\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \times \iota_C) = \pi_A \circ g \\
 & \wedge \\
 & \text{EV}_{BC} \circ (p_{B^C} \times \iota_C) \circ (\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle \times \iota_C) = \pi_B \circ g \\
\equiv & \{\text{Properties composition and product map}\} \\
 & \text{EV}_{AC} \circ ((p_{A^C} \circ \langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle) \times \iota_C) = \pi_A \circ g \\
 & \wedge \\
 & \text{EV}_{BC} \circ ((p_{B^C} \circ \langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle) \times \iota_C) = \pi_B \circ g \\
\equiv & \{\text{UMP } \langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle\} \\
 & \text{EV}_{AC} \circ (\text{CURRY}(\pi_A \circ g) \times \iota_C) = \pi_A \circ g \\
 & \wedge \\
 & \text{EV}_{BC} \circ (\text{CURRY}(\pi_B \circ g) \times \iota_C) = \pi_B \circ g \\
\equiv & \{\text{UMP } \text{CURRY}(\pi_A \circ g) \text{ and } \text{CURRY}(\pi_B \circ g)\} \\
 & \mathit{true} \wedge \mathit{true}
\end{array}$$

We have left to prove that $\langle \text{CURRY}(\pi_A \circ g), \text{CURRY}(\pi_B \circ g) \rangle$ is the unique arrow u such that $\text{EV}_{new} \circ (u \times \iota_C) = g$. The argument is similar to the previous one.

$$
\begin{aligned}
& \textsc{ev}_{new} \circ (u \times \iota_C) = g \\
\equiv \;& \{\text{Composition with projections and Lemma 5.9}\} \\
& \pi_A \circ \textsc{ev}_{new} \circ (u \times \iota_C) = \pi_A \circ g \\
& \wedge \\
& \pi_B \circ \textsc{ev}_{new} \circ (u \times \iota_C) = \pi_B \circ g \\
\equiv \;& \{\text{UMP } \textsc{ev}_{new}\} \\
& \textsc{ev}_{AC} \circ (p_{A^C} \times \iota_C) \circ (u \times \iota_C) = \pi_A \circ g \\
& \wedge \\
& \textsc{ev}_{BC} \circ (p_{B^C} \times \iota_C) \circ (u \times \iota_C) = \pi_B \circ g \\
\equiv \;& \{\text{Properties composition and product map}\} \\
& \textsc{ev}_{AC} \circ ((p_{A^C} \circ u) \times \iota_C) = \pi_A \circ g \\
& \wedge \\
& \textsc{ev}_{BC} \circ ((p_{B^C} \circ u) \times \iota_C) = \pi_B \circ g \\
\equiv \;& \{\text{UMP } \textsc{curry}(\pi_A \circ g) \text{ and } \textsc{curry}(\pi_B \circ g)\} \\
& p_{A^C} \circ u = \textsc{curry}(\pi_A \circ g) \\
& \wedge \\
& p_{B^C} \circ u = \textsc{curry}(\pi_B \circ g) \\
\equiv \;& \{\text{UMP } \langle\, \textsc{curry}(\pi_A \circ g), \textsc{curry}(\pi_B \circ g)\,\rangle\} \\
& u = \langle\, \textsc{curry}(\pi_A \circ g), \textsc{curry}(\pi_B \circ g)\,\rangle
\end{aligned}
$$

□

Lemma 7.5. Let **C** be a CCC, and let B^A be an exponential object, with evaluation arrow $\textsc{ev} : B^A \times A \longrightarrow B$. The arrow $\textsc{curry}(\textsc{ev})$ is the identity $\iota_{B^A \times A}$.

Proof. On the one hand, we know that

$$h = \textsc{curry}(g) \equiv g = \textsc{ev} \circ (h \times \iota_A)$$

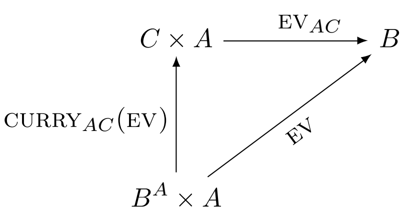
On the other hand we know that

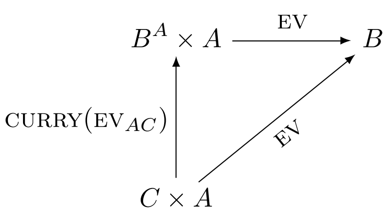
$$\textsc{ev} = \textsc{ev} \circ \iota_{B^A \times A} = \textsc{ev} \circ (\iota_{B^A} \times \iota_A)$$

It follows that $\iota_{B^A} = \textsc{curry}(\textsc{ev})$. □

Proposition 7.6. Let **C** be a CCC, and C be an exponential of an object B with respect to A. The objects C and B^A are isomorphic.

Proof. We have the following diagrams

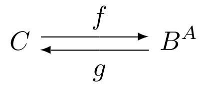




with the UMP's

$$h = \textsc{curry}_{AC}(g) \equiv g = \textsc{ev}_{AC} \circ (h \times \iota_A)$$
$$h = \textsc{curry}(g) \equiv g = \textsc{ev} \circ (h \times \iota_A)$$

We have to prove that there exists two arrows



such that $f \circ g = \iota_{B^A}$ and $g \circ f = \iota_C$. Consider the following

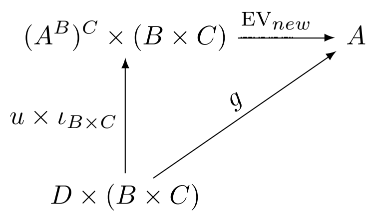
$$
\begin{aligned}
& true \\
\equiv\ & \{\text{UMP } \textsc{curry}_{AC}(\textsc{ev})\} \\
& \textsc{ev}_{AC} \circ (\textsc{curry}_{AC}(\textsc{ev}) \times \iota_A) = \textsc{ev} \\
\equiv\ & \{\text{Axiom of composition}\} \\
& \textsc{ev}_{AC} \circ (\textsc{curry}_{AC}(\textsc{ev}) \times \iota_A) \circ (\textsc{curry}(\textsc{ev}_{AC}) \times \iota_A) = \textsc{ev} \circ (\textsc{curry}(\textsc{ev}_{AC}) \times \iota_A) \\
\equiv\ & \{\text{Properties composition and product map}\} \\
& \textsc{ev}_{AC} \circ ((\textsc{curry}_{AC}(\textsc{ev}) \circ \textsc{curry}(\textsc{ev}_{AC})) \times \iota_A) = \textsc{ev} \circ (\textsc{curry}(\textsc{ev}_{AC}) \times \iota_A) \\
\equiv\ & \{\text{UMP } \textsc{curry}(\textsc{ev}_{AC})\} \\
& \textsc{ev}_{AC} \circ ((\textsc{curry}_{AC}(\textsc{ev}) \circ \textsc{curry}(\textsc{ev}_{AC})) \times \iota_A) = \textsc{ev}_{AC} \\
\equiv\ & \{\text{UMP } \textsc{curry}_{AC}(\textsc{ev})\} \\
& (\textsc{curry}_{AC}(\textsc{ev}) \circ \textsc{curry}(\textsc{ev}_{AC})) = \textsc{curry}_{AC}(\textsc{ev}_{AC}) \\
\equiv\ & \{\text{Previous lemma}\} \\
& \textsc{curry}_{AC}(\textsc{ev}) \circ \textsc{curry}(\textsc{ev}_{AC}) = \iota_{B^A}
\end{aligned}
$$

An analogous argument let us prove that $\textsc{curry}(\textsc{ev}_{AC}) \circ \textsc{curry}_{AC}(\textsc{ev}) = \iota_C$.

□

Proposition 7.7. In a CCC the object $(A^B)^C$ is an exponential of A with respect to $B \times C$.

Proof. We have to prove three things; that 1) there exists an arrow $\textsc{ev}_{new} : (A^B)^C \times (B \times C) \longrightarrow A$ such that 2) for every arrow $g : D \times (B \times C) \longrightarrow A$ there exist an arrow $u : D \longrightarrow (A^B)^C$ which makes the diagram

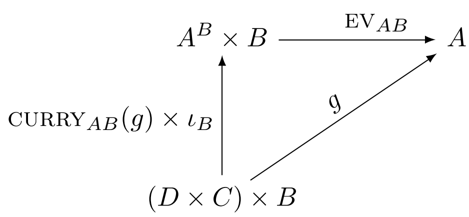


commute, and 3) that is unique. We prove 1) and 2).

As the products are associative and commutative, when useful we will drop or move the parenthesis, and we will reason up to the isomorphisms

$$D \times (C \times B) \cong (D \times C) \times B \cong D \times (B \times C)$$

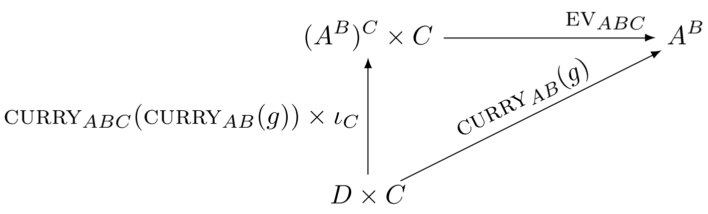
Consider an object D with an arrow $g : D \times (B \times C) \longrightarrow A$; since we are in a CCC, there exists the exponential A^B, thus we have the diagram



with

$$h = \textsc{curry}_{AB}(g) \equiv g = \textsc{ev}_{AB} \circ (h \times \iota_B)$$

We have now the arrow $\textsc{curry}_{AB}(g) : D \times C \longrightarrow A^B$. We use again the hypothesis of being in a CCC, to use the exponential object $(A^B)^C$. Since it exists, the following diagram commute



with $\textsc{curry}_{ABC}(\textsc{curry}_{AB}(g)) : D \longrightarrow (A^B)^C$ and

$$h = \textsc{curry}_{ABC}(g) \equiv g = \textsc{ev}_{ABC} \circ (h \times \iota_C)$$

Thanks to product maps, we have the arrows

$$(A^B)^C \times C \times B \xrightarrow{\textsc{ev}_{ABC} \times \iota_B} A^B \times B \xrightarrow{\textsc{ev}_{AB}} A$$

Let $\textsc{ev}_{new} = \textsc{ev}_{AB} \circ (\textsc{ev}_{ABC} \times \iota_B)$. We prove that $u = \textsc{curry}_{ABC}(\textsc{curry}_{AB}(g))$

$$\begin{array}{ll}
 & g = \textsc{ev}_{new} \circ (\textsc{curry}_{ABC}(\textsc{curry}_{AB}(g)) \times \iota_{C \times B}) \\
\equiv & \{\text{Identity of produts is the product map of the identities}\} \\
 & g = \textsc{ev}_{new} \circ (\textsc{curry}_{ABC}(\textsc{curry}_{AB}(g)) \times (\iota_C \times \iota_B)) \\
\equiv & \{\text{Associativity product maps}\} \\
 & g = \textsc{ev}_{new} \circ ((\textsc{curry}_{ABC}(\textsc{curry}_{AB}(g)) \times \iota_C) \times \iota_B) \\
\equiv & \{\text{Definition of } \textsc{ev}_{new}\} \\
 & g = \textsc{ev}_{AB} \circ (\textsc{ev}_{ABC} \times \iota_B) \circ ((\textsc{curry}_{ABC}(\textsc{curry}_{AB}(g)) \times \iota_C) \times \iota_B) \\
\equiv & \{\text{Properties composition and product maps}\} \\
 & g = \textsc{ev}_{AB} \circ (\textsc{ev}_{ABC} \circ (\textsc{curry}_{ABC}(\textsc{curry}_{AB}(g)) \times \iota_C) \times (\iota_B \circ \iota_B)) \\
\equiv & \{\text{UMP } \textsc{curry}_{ABC}(\textsc{curry}_{AB}(g))\} \\
 & g = \textsc{ev}_{AB} \circ (\textsc{curry}_{AB}(g) \times (\iota_B \circ \iota_B)) \\
\equiv & \{\text{Identity axiom}\} \\
 & g = \textsc{ev}_{AB} \circ (\textsc{curry}_{AB}(g) \times \iota_B) \\
\equiv & \{\text{UMP } \textsc{curry}_{AB}(g)\} \\
 & \mathit{true}
\end{array}$$

We have proven 1) and 2). □

Exercise 7.8.

no. **6.8** (**2**) in **[Awo10]**

no. **1.10.5** (**5**) in **[Pie91]**

Show that for any three objects A, B, C in a CCC, there are the isomorphisms:

(a) $(A \times B)^C \cong A^C \times B^C$

(b) $(A^B)^C \cong A^{B \times C}$

Proof. Point (a) follows from Proposition 7.4, and Proposition 7.6. The proof of point (b) follows from Proposition 7.7 and Proposition 7.6. □

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