1. Let X be the number that comes up on a fair 6-sided die. What is E[X] and Var(X) ? Let I be a random variable which takes value 1 when the die value is greater than 3 and I equals 0 otherwise. What is E[I] and Var(I) ? Now suppose we throw 10 dice. Let X_i be the result for i^{th} die. What is $E[X_1+X_2+...+X_{10}]$?

<u>Solution</u>

(i) E[X] = 1x1/6 + 2x1/6+3x1/6+4x1/6+5x1/6+6x1/6=3.5

(ii) $E[X^2] = 1^2x1/6 + 2^2x1/6+3^2x1/6+4^2x1/6+5^2x1/6+6^2x1/6=91/6=15.17$

$Var(X) = E[X^2]-E[X]^2 = 15.17-3.5^2 = 2.92$

(iii) Sample space of die is {1,2,3,4,5,6}. Event that greater than 3 is {4,5,6}. So P(I=1)=3/6=1/2 and P(I=0)=1-P(I=1)=1/2

E[I] = 1x1/2+0x1/2 = 1/2 (which equals P(I=1) as we know since I is an indicator variable)

(iv) $E[I^2] = 1^2x1/2 + 0^2$ x 1/2 =1/2

$Var(I) = E[X^2]-E[X]^2 =1/2-(1/2)^2=1/2-1/4=1/4$

(v) Using the linearity of the expectation,

$E[X_1+X_2+...+X_{10}] = E[X_1]+E[X_2]+...+E[X_{10}]=10x3.5=35$

2. A computer program crashes at the end of each hour with probability p, if has not done so already. What is the expected time until the program crashes ? Express in terms of p. Useful fact[1]: $\sum_{i=1}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$. Write a Matlab simulation and compare its estimate with your calculation.

<u>Solution</u>

Let X be the number of hours until program crashes.

$P(X=1)=p$

$P(X=2)=p(1-p)$ (did not crash in first hour but did crash in second)

$P(X=3)=p(1-p)^2$

and so on. So,

$$E[X] = \sum_{i=1}^{\infty} iP(X=i) = \sum_{i=1}^{\infty} ip(1-p)^{(i-1)} = p\sum_{i=1}^{\infty} i(1-p)^{(i-1)} = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

[1] E.g. see https://en.wikipedia.org/wiki/Arithmetico-geometric_sequence

Matlab:

```
P=[]; p=0.1; N=10000;
for i=1:N,
    count = 1;
    while (rand>p), count=count+1; end
    P=[P;count];
end
sum(P)/N
```

3. The time to run a cloud computing task is X = 1/N + Y, where N is the number of servers allocated to the task and Y is a random variable with E[Y]=0. Suppose n=2 servers are allocated, what is E[X|N=2] ? Suppose the number of available servers has PMF: P(N=1)=0.5, P(N=2)=0.2, P(N=3)=0.2, P(N=4)=0.1. What is E[X] ?

Solution

(i) E[X|N=2] = E[1/2 + Y] = 1/2+E[Y]=1/2

(ii) E[X] = E[X|N=1]P(N=1)+ E[X|N=2]P(N=2)+ E[X|N=3]P(N=3)+ E[X|N=4]P(N=5) = 1x0.5 + 1/2x0.2+1/3x0.2+1/4x0.1 = 0.69

4. Suppose the number of people using a mobile app in a day is a random variable N with PMF P(N=n) =$\frac{e^{-n}}{1-1/e}$. On average each person pays €1 per day to use the service. What is the expected revenue for this app ? Useful fact: $\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$

Solution

Let X_i be the revenue from user i, E[X_i]=1. Let $Y = \sum_{i=1}^{N} X_i$ be the revenue in a day.

$$E[Y] = \sum_{n=0}^{\infty} E[Y|N = n]P(N = n)$$

Now $E[Y|X = n] = E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = n$

So $E[Y] = \sum_{n=0}^{\infty} n\frac{e^{-n}}{1-1/e}$=$(\frac{1}{1-1/e}) \sum_{n=1}^{\infty} ne^{-n}$=$\left(\frac{1}{1-1/e}\right)\frac{\frac{1}{e}}{(1-1/e)^2} = 1.46$

5. A tout is buying tickets for a concert. The price of the tickets is €50 and the tout sells them for €100, making a profit of €50. However, if he can't sell a ticket then he makes nothing from it (so it costs him €50). Let random variable N be the number of people who will buy tickets from the tout. N has PMF P(N=n)= $(1-e^{-0.5})e^{-0.5n}$. Suppose the tout buys m=10 tickets, what is his expected profit or loss ? Express in terms of m, n and constants. Using Matlab plot the expected profit vs m. What value of m maximizes profit ?

Solution

Let random variable Y be the tout's revenue.

$$E[Y] = \sum_{n=0}^{\infty} E[Y|N=n]P(N=n)$$

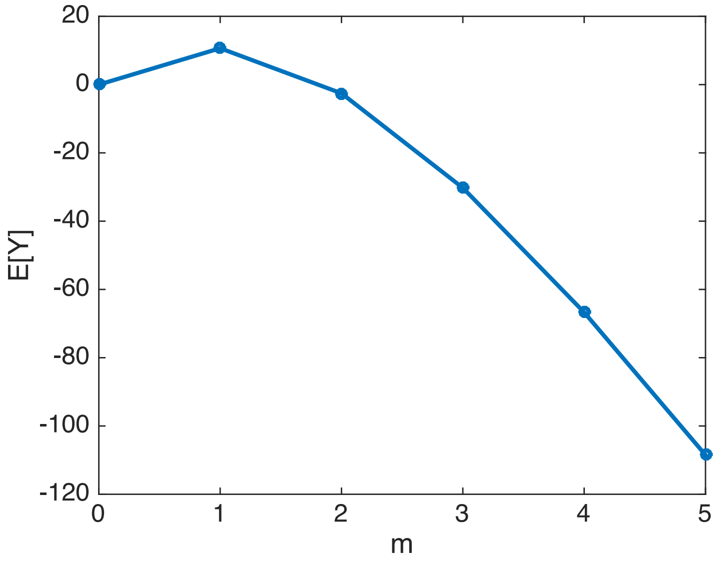
Now $E[Y|N=n] = \begin{cases} 50n - 50(m-n) & if\ n < m \\ 50m & if\ n \geq m \end{cases}$. So,

$$E[Y] = 50(1-e^{-0.5})(\sum_{n=0}^{m-1}\left(n-(m-n)\right)e^{-0.5n} + \sum_{n=m}^{\infty} m\, e^{-0.5n})$$

Matlab:

```
Y=[];

for m=0:5,

        y=0;

        for n=0:(m-1), y=y+(n- (m-n))*exp(-0.5*n);end;

        for n=m:100, y=y+m*exp(-0.5*n);end;

        Y=[Y; m, y*50*(1-exp(-0.5))];

end

plot(Y(:,1),Y(:,2))
```

Maximum expected profit when m=1, when E[Y]=€10.65



20
0
-20
-40
-60
-80
-100
-120
E[Y]
0
1
2
3
4
5
m