1. Suppose that X and Y are independent random variables and have PMF P(X=1)=P(Y=1)=0.1, P(X=2)=P(Y=2)=0.2, P(X=3)=P(Y=3)=0.3, P(X=4)=P(Y=4)=0.4. Compute P(X+Y≥7). Write a Matlab simulation and compare its results against your calculations.

Solution

(i) $P(X+Y \geq 7) = P(X=3 \text{ and } Y=4) + P(X=4 \text{ and } Y=3) + P(X=4 \text{ and } Y=4)$

$= P(X=3)P(Y=4) + P(X=4)P(Y=3) + P(X=4)P(Y=4)$

$= 0.3 \times 0.4 + 0.4 \times 0.3 + 0.4 \times 0.4 = 0.4$

(ii) Matlab:

```
P=[]; N=10000;

for i=1:N,

        cdf = [0,0.1, 0.1+0.2, 0.1+0.2+0.3];

        X= max(find(cdf<=rand));

        Y = max(find(cdf<=rand));

        P=[P; X+Y>=7];

end

sum(P)/N
```

2. A bit string of length 10 is sent across a lossy link. Each bit is corrupted independently with probability 0.1. What is the probability that there are at least 3 bit errors ? Write a Matlab simulation and check its results against your calculations.

Solution

Let X be the number of errors. $P(X=0)= (1-0.1)^{10}$, $P(X=1)= \binom{10}{1}0.1(1-0.1)^9$, $P(X=2)=\binom{10}{2}0.1^2(1-0.1)^8$. $P(X\geq3) = 1-(P(X=0)+P(X=1)+P(X=2)) = 0.0702$.

Matlab:

```
P=[]; N=10000;

for i=1:N,

        errors = rand(1,10)<=0.1;

        P=[P;sum(errors)>=3];

end

sum(P)/N
```

3. Consider a computer that has two operating systems installed on it. Let X and Y be the number of times the computer freezes in a day when it runs on the first and second operating systems respectively. The following reports the probability of different numbers of freezes.

	y=0	y=1	y=2
x=0	0.5	0.05	0.12
x=1	0.10	0.07	0.01
x=2	0.08	0.06	0.01

Are X and Y independent ? Let Z=XY. Find the PMF of Z.

Solution

(i) X and Y are independent if P(X=x and Y=y)=P(X=x)P(Y=y) for all x and y.

P(X=0) = P(X=0 and Y=0) + P(X=0 and |Y=1) + P(X=0 and |Y=2) = 0.5+0.05+0.12=0.67

P(X=1) = 0.10+0.07+0.01 = 0.18

P(X=2) = 0.08+0.06+0.01 = 0.15

P(Y=0) = 0.5+0.1+0.08 = 0.68

P(Y=1) = 0.05+0.07+0.06 = 0.18

$P(Y=2) = 0.12+0.0+0.01 = 0.14$

$P(X=0 \text{ and } Y=0) = 0.5$ but $P(X=0)P(Y=0) = 0.67 \times 0.68 = 0.455$ so X and Y are not independent.

(ii) X takes values 0,1 or 2 and Y takes values 0,1 or 2. So Z takes values 0,1,2 ,4.

$P(Z=0) = 0.5+0.1+0.08+0.05+0.12 = 0.85$

$P(Z=1) = 0.07$

$P(Z=2) = 0.06+0.01 = 0.07$

$P(Z=4) = 0.01$

4. Suppose 5% of coins are not fair, with probability 0.2 of coming up heads. We toss a coin 10 times and observe less than 3 heads. What is the probability that the coin is not fair ? Suppose that only 1% of coins are not fair, what is the probability now ?

Solution

(i) Let X be the number of heads and let indicator variable Y=1 when the coin is not fair and 0 otherwise. By Bayes Rule:

$P(Y=1|X<3) = P(X<3|Y=1)P(Y=1)/P(X<3)$

$P(Y=1) = 0.05$ (5% of coins are not fair)

$P(X<3|Y=1) = P(X=0|Y=1) + P(X=1|Y=1) + P(X=2|Y=1) = (1-0.2)^{10}+ \binom{10}{1}0.2(1-0.2)^9+ \binom{10}{2}0.2^2(1-0.2)^8 = 0.6778$

$P(X<3|Y=0) = (1-0.5)^{10}+ \binom{10}{1}0.5(1-0.5)^9+ \binom{10}{2}0.5^2(1-0.5)^8 = 0.0547$

$P(X<3) = P(X<3|Y=1)P(Y=1) + P(X<3|Y=0)(1-P(Y=1)) = 0.6778 \times 0.05+0.0547 \times 0.95 = 0.0859$

So $P(Y=1|X<3) = 0.6778 \times 0.05/0.0859 = 0.394$

(ii) When $P(Y=1) = 0.01$ (1% of coins are not fair) then

$P(X<3) = 0.6778 \times 0.01+0.0547 \times 0.99 = 0.0609$

$P(Y=1|X<3) = 0.6778 \times 0.01/0.0609 = 0.111$