1. Roll a fair 6-sided die twice. What is the probability that we roll (i) two of the *same* numbers, (ii) two *different* numbers ?

<u>Solution</u>

Sample space S={(1,1),(1,2),(1,3), … (6,1),(6,2),…(6,6)}. Size |S|=36. Probability that we roll two 1's is 1/36, probability that we roll two 2's is 1/36, etc. There are 6 different numbers, so we can roll the same number twice with probability 6x1/36 = 1/6. Probability that we roll two different numbers is 1-1/6=5/6.

2. Willy Wonka issues 7 golden tickets in a supply of n chocolate bars. If I buy k chocolate bars, what is the probability that I find a golden ticket ? Express your answer in terms of k, n and constants.

<u>Solution</u>

We start by calculating the probability that none of the k bars that I buy contain a golden ticket. If I pick one chocolate bar, the probability that it does not contain a golden ticket is $1\text{-}7/n$. Assuming the first bar did not have a golden ticket I now pick a second bar from the remaining n-1 bars. The probability that it does not contain a golden ticket is $1\text{-}7/(n\text{-}1)$. Repeating for k bars, the probability that none of the k bars contain a golden ticket is $p=(1-7/n)(1-7/(n-1))\ldots(1-7/(n-k+1))$. And so the probability that at least one contains a golden ticket is $1\text{-}p$.

3. Instead of splitting the room cleaning duties, you and your roommate both roll a die and decide to let the person with the highest die roll clean everything. For example, if you roll a 3 and your friend rolls a 6 then your roommate cleans the room. You, being the bigger person, will clean the room should there be a tie. What is the probability that you clean the room ?

<u>Solution 1</u>

Sample space S={(1,1),(1,2),(1,3), … (6,1),(6,2),…(6,6)}, |S|=36.

Let F be the event that you clean the room. This is {(1,1),(1,2),…,(1,6), (2,2),(2,3),…,(2,6),(3,3),(3,4),…,(3,6),(4,4),(4,5),(4,6),(5,5),(5,6),(6,6)}.
|F|=6+5+4+3+2+1=21.

So P(F)=21/36 = 0.58

<u>Solution 2</u>

Sample space S={(1,1),(1,2),(1,3), … (6,1),(6,2),…(6,6)}. Let F be the event that you clean the room.

Event E_1: Your friend rolls a 1. Then you always clean the room, $P(F|E_1)=1$.

Event E_2: Your friend rolls a 2. With probability 5/6 you clean the room (you throw a 2,3,4,5 or 6). $P(F|E_2)=5/6$

Event E_3: Your friend rolls a 3, then with probability 4/6 you clean room. $P(F|E_3)=4/6$

And so on.

Now $P(F) = P(F\cap E_1) + P(F\cap E_2) + \ldots + P(F\cap E_6)$ since E_1, E_2, etc are mutually exclusive events. That is,

$P(F) = P(F|E_1)P(E_1) + P(F|E_2)P(E_2) + \ldots + P(F|E_6)P(E_6)$

$= 1\times\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}+\frac{4}{6}\times\frac{1}{6}+\cdots+\frac{1}{6}\times\frac{1}{6} = 0.58$

4. Consider a population where 30% of people suffer from a certain disease. There is an imperfect test for detecting the disease. When applied to a person with the disease the test gives a positive result 95% of the time. When applied to a person who does not have the disease, the test gives a negative result 95% of the time. Suppose that the test is positive for a person. What is the probability that the person has the disease ?

Solution

Apply Bayes Rule. Let E be the event that the person has the disease and F be the event that the test is positive. We have:

$P(E|F) = P(F|E)P(E)/P(F)$

Now,

$P(E)=0.3$ (30% of the population have the disease)

$P(F|E) = 0.95$ (if have the disease, test is positive 95% of the time)

$P(F| E^c) = 0.05$ (if don't have the disease, test is positive 5% of the time)

$P(E^c) = 1-P(E)=0.7$

$P(F)=P(F|E)P(E) + P(F|E^c)P(E^c) = 0.95x0.3+0.05x0.7 = 0.32$

So $P(E|F) = 0.95x0.3/0.32 = 0.89$

5. A motorway bridge uses cameras to read car number plates and charge tolls. If a person has enough funds in their account to pay the toll, this is correctly noted 99% of the time and the charge deducted. Otherwise, the camera correctly detects non-payment 99% of the time and issues a penalty notice. Suppose 1% of the cars passing over the bridge do not have sufficient funds. What is the probability that a person who receives a penalty notice in fact has sufficient funds i.e. that the penalty notice has been sent incorrectly ?

Solution

Let E be the event that a person has sufficient funds and F be the event that they receive a penalty notice. We have:

$P(E|F) = P(F|E)P(E)/P(F)$

Now,

$P(E)=0.99$ (1% of cars have insufficient funds, so 99% have sufficient funds)

$P(F|E) = 0.01$ (if have sufficient funds, a notice is incorrectly sent 1% of the time)

$P(F| E^c) = 0.99$ (if don't sufficient funds, this is correctly noted 99% of the time)

$P(E^c) = 1-P(E)=0.01$

$P(F)=P(F|E)P(E) + P(F|E^c)P(E^c) = 0.01x0.99+0.99x0.01 = 0.0198$

So $P(E|F) = 0.01x0.99/0.0198 = 0.5$

6. Suppose two websites A and B rent books. Site A receives 60% of all orders and site B 40%. Among the orders placed at site A, 75% arrive on time. Among the orders placed at site B, 90% arrive on time. Given that an order arrived on time, what is the probability that the order was placed on site B ?

Solution

Let E be the event that the order was placed on site B and F be the event that the book arrives on time. By Bayes Rule,

$P(E|F) = P(F|E)P(E)/P(F)$

Now,

$P(E) = 0.4$ (40% of orders are placed on site B)

$P(F|E) = 0.9$ (90% of books ordered on site B arrive on time)

$P(F| E^c) = 0.75$ (75% of books ordered at site B arrive on time)

$P(E^c) = 1-P(E)=0.6$

$P(F)=P(F|E)P(E) + P(F|E^c)P(E^c) = 0.9x0.4 + 0.75 x 0.6 = 0.81$

So $P(E|F) = 0.9x0.4/0.81 = 0.44$

7. Suppose 1% of the population are gifted with super powers. You have just noticed that you might possess a super power. Assuming you do indeed possess a super power, you correctly observe its effects with probability 0.8, otherwise you mistake its effects as coincidence. Assuming you do not possess a super

power, you correctly observe this with probability 0.99. What is the probability that you possess a super power ?

Solution

Let E be the event that you possess a superpower and F be the event that you observe its effects. By Bayes Rule,

$P(E|F) = P(F|E)P(E)/P(F)$

Now,

$P(E) = 0.01$ (1% of people have superpowers)

$P(F|E) = 0.8$ (correctly observe its effects 80% of the time)

$P(F| E^c) = 0.01$ (mistake normality for super power 1% of the time)

$P(E^c) = 1-P(E)=0.99$

$P(F)=P(F|E)P(E) + P(F|E^c)P(E^c) = 0.8x0.01 + 0.01 \times 0.99 = 0.0179$

So $P(E|F) = 0.8x0.01/0.0179 = 0.44$