Overview

Recall module is roughly split into four parts:

1. Random events: counting, events, axioms of probability, Bayes, independence
2. Random variables: discrete RVs, mean and variance, correlation, conditional expectation
   Mid-term
3. Inequalities and laws of large numbers: Markov, Chebyshev, Chernoff bounds, sample mean, weak law of large numbers, central limit theorem, confidence intervals, bootstrapping
4. Statistical models: continuous random variables, logistic regression, least squares

Overview

- Markov's Inequality
- Chebyshev's Inequality
- Chernoff's Inequality

Why Are Inequalities Useful ?

We may not know the true form of a probability distribution

- Opinion polls

  > The 2016 General Election Exit Poll was conducted exclusively on behalf of *The Irish Times* by Ipsos MRBI, among a national sample of 5,260 voters at 200 polling stations throughout all constituencies in the Republic of Ireland.
  >
  > Voters were randomly selected to self-complete a mock ballot paper on exiting the polling station. The accuracy level is estimated to be approximately plus or minus 1.2 per cent.

- Stock market data
- Weather tomorrow

But we may know some of its properties

- Mean
- Variance
- Non-negativity

Inequalities let us say something about the probability distribution in such cases, although often imprecise. They are also v important for looking at what happens as we collect more and more measurements.

Markov's Inequality

Often we want to know:

What is the probability that the value of r.v. X is "far" from its mean ?

A generic answer for non-negative X is Markov's inequality. Say X is a non-negative random variable. Then:

$$P(X \geq a) \leq \frac{E(X)}{a} \text{ for all } a > 0$$

Proof:

- Let indicator $I_a(X) = 1$ if $X \geq a$ and $I_a(X) = 0$ otherwise. Then $aI_a(X) \leq X$ i.e. $I_a(X) \leq \frac{X}{a}$.
- $E(I_a(X)) \leq E(\frac{X}{a}) = \frac{E(X)}{a}$
- $E(I_a(X)) = P(X \geq a) \leq \frac{E(X)}{a}$

Markov's Inequality

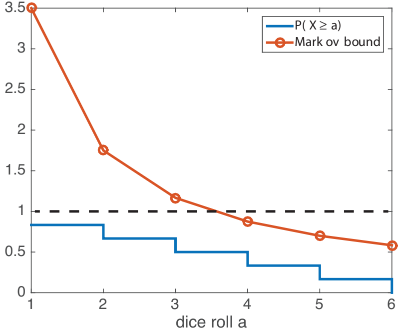
Andrey Andreyevich Markov (1856-1922) was a Russian mathematician

- Markov's inequality is named after him
- Also Markov Chains, used e.g. in Google's PageRank algorithm

Markov's Inequality

Example: Roll 6-sided dice.

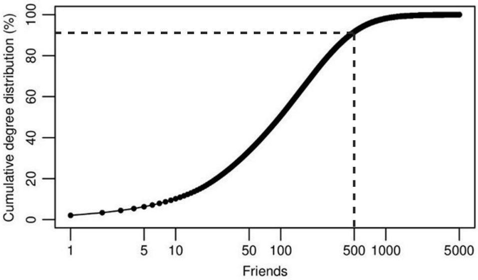
- Mean is $E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5$
- Markov inequality: $P(X \geq 5) \leq \frac{3.5}{5} = 0.7$. Exact: $P(X = 5) + P(X = 6) = \frac{1}{6} + \frac{1}{6} = 0.33$
- So a loose bound, but it made no assumptions about the form of distribution.



Markov's Inequality

Example: Distribution of number X of facebook friends.

- Mean is $E(X) = 190$ (!)
- Markov inequality: $P(X \geq 500) \leq \frac{190}{500} = 0.38$. From plot, $P(X \geq 500) \approx 0.1$.
- Markov inequality: $P(X \geq 190) \leq \frac{190}{190} = 1$, non-informative. From plot, $P(X \geq 190) \approx 0.3$.

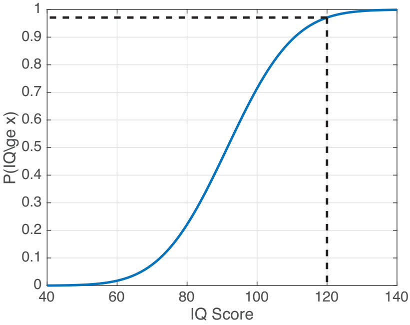


source: http://arxiv.org/abs/1111.4503

Markov's Inequality

Example: IQ in Ireland

- Mean is $E(X) = 92$. Score 110-119 = "high average", 120-129="superior".
- Markov inequality: $P(X \geq 110) \leq \frac{92}{110} = 0.83$. From data, $P(X \geq 110) \approx 0.11$.
- Markov inequality: $P(X \geq 120) \leq \frac{92}{120} = 0.76$. From data, $P(X \geq 120) \approx 0.029$.



Chebyshev's Inequality

Suppose X is a random variable with mean $E(X) = \mu$ and variance $var(X) = \sigma^2$. Then

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2} \text{ for all } k > 0$$

Proof:

- Since $(X - \mu)^2$ is a non-negative random variable we can apply Markov's inequality with $a = k^2$ to get

$$P((X - \mu)^2 \geq k^2) \leq \frac{E((X - \mu)^2)}{k^2} = \frac{\sigma^2}{k^2}$$

- Note that $(X - \mu)^2 \geq k^2 \iff |X - \mu| \geq k$, so

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

Chebyshev's Inequality

Pafnuty Lvovich Chebyshev (1821 - 1894) also a Russian mathematician

- Chebyshev's inequality was in fact first formulated by French mathematician Jules Bienaymé without proof, then proved by Chebyshev 14 years later.
- Markov was a graduate student of Chebyshev (also Aleksandr Lyapunov, but that's another days work)

Chebyshev's Inequality

Chebyshev's inequality links the "spread" of values of a random variable around its mean to the variance σ^2:

- Applying Chebyshev's inequality $P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$ with $k = n\sigma$ gives:

$$P(|X-\mu| \geq n\sigma) \leq \frac{1}{n^2}$$

- With $n = 3$ then $P(|X-\mu| \geq 3\sigma) \leq \frac{1}{9} = 0.11$.
- This holds even when distribution is not Gaussian, so can be quite handy (if conservative).

Chebyshev's Inequality

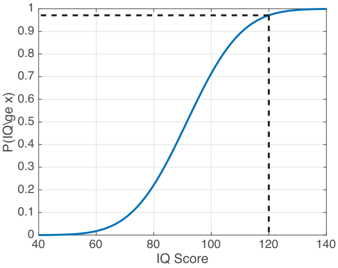
Example: Roll 6-sided dice.

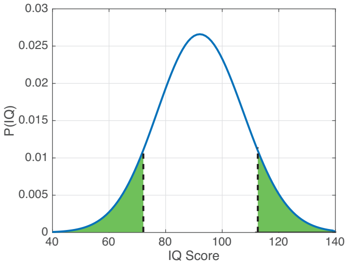
- Mean is $E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5$
- Variance is $Var(X) = E[X^2] - E[X]^2$.
  $E[X^2] = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + \cdots + 6^2 \times \frac{1}{6} \approx 15.17$,
  $Var(X) = 15.17 - 3.5^2 \approx 2.9$
- Chebyshev inequality: $P(|X - 3.5| \geq 2.5) \leq \frac{2.9}{2.5^2} = 0.46$.
- Exact: $P(|X - 3.5| \geq 2.5) = P(X = 1) + P(X = 6) = \frac{1}{6} + \frac{1}{6} = 0.33$
- A loose bound, but use of variance in Chebyshev inequality can improve accuracy cf Markov inequality.

Chebyshev''s Inequality

Example: IQ in Ireland

- Mean is $E(X) = 92$, variance is $\sigma^2 = 225$.
- Chebyshev inequality: $P(|X - 92| \geq 20) \leq \frac{225}{400} = 0.56$. From data, $P(|X - 92| \geq 20) \approx 0.18$.
- Markov inequality: $P(X \geq 112) \leq \frac{92}{112} = 0.82$. And need to add $P(X \leq 72)$ to this.





Chernoff's Inequality

Suppose we have random variable X. Then

$$P(X \geq a) \leq \min_{t>0} e^{-ta} e^{\log E(e^{tX})}$$

Proof:

- $P(X \geq a) = P(e^{tX} \geq e^{ta})$ for $t > 0$.
- By Markov's inequality

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq \frac{E(e^{tX})}{e^{ta}} = e^{-ta} E(e^{tX})$$

- This holds for all $t > 0$, so might as well choose the one that minimises it.

Chernoff's Inequality

Herman Chernoff is a US mathematician (with Russian parents)

- Was at MIT, then Harvard.

Chernoff's Inequality

- $P(X \geq a) \leq \min_{t>0} e^{-ta} e^{\log E(e^{tX})}$
- $E(e^{tX})$ is called the "moment generating function"
- Contains more information about the distribution than just the mean (used by Markov inequality) and variance (used by Chebyshev inequality).

Chernoff's Inequality

Example: coin flipping:

- A fair coin lands on heads with probability $1/2$ and on tails with probability $1/2$.
- If coin is flipped 100 times, give an upper bound on the probability that it lands heads at least 60 times.
- Random variable $X_k = 1$ is heads, $X = 0$ if tails at flip k. $S = \sum_{k=1}^{100} X_k$.
- $E(e^{tX_k}) = \frac{1}{2}e^{t\times 1} + \frac{1}{2}e^{t\times 0} = \frac{1}{2}(e^t + 1)$

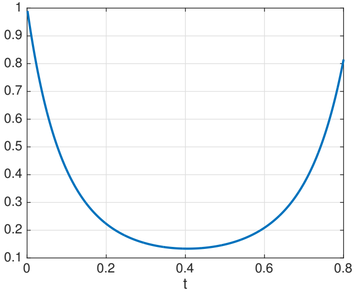
$$\log E(e^{tS}) = \log E(\prod_{k=1}^{100} e^{tX_k}) = \log \prod_{k=1}^{100} E(e^{tX_k}) = 100 \log(\frac{1}{2}(e^t + 1))$$

- By Chernoff's inequality, the probability of at least 55 heads is

$$P(S \geq 60) \leq \min_{t>0} e^{-60t} e^{\log E(e^{tX})} = \min_{t>0} e^{-60t} e^{100 \log(\frac{1}{2}(e^t+1))}$$

Chernoff's Inequality

- $P(S \geq 60) \leq \min_{t>0} e^{-60t} e^{100 \log(\frac{1}{2}(e^t+1))}$

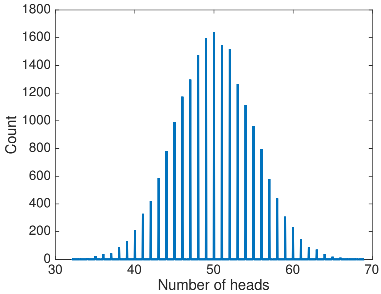


- Using $t = 0.4$, Chernoff's inequality gives $P(S \geq 60) \leq 0.13$.
- Markov inequality gives $P(S \geq 60) \leq \frac{E(S)}{60} = \frac{50}{60} = 0.83$

Chernoff's Inequality

- Let's try it ...

```
XX=[];
for i=1:20000,
XX=[XX, sum(rand(1,100)<0.5)];
end
[n,x]=hist(XX,1000);
plot(x,n)
xlabel('Number of heads')
ylabel('Count')
```



- Out of 20,000 trials 592 have ≥ 60 heads i.e 0.03. Cf Markov inequality value of 0.83 and Chernoff inequality value of 0.13.