Overview

- Notation
- Sample Spaces
- Events
- Set Operations
- Axioms of Probability

Notation

- $\mathbb{Z}$ is the integers
- $\mathbb{R}$ is the real numbers
- $\{\cdots\}$ is a set
- $A \subset B$ means set A is a subset of set B
- $A \in B$ means A is a member of set B
- $\emptyset$ is the empty set
- $|A|$ is the number of elements in set A
- $|$ means "such that
  e.g. $\{2z|z \in \{1, 2, 3\}\} = \{2, 4, 6\}$
- $P(E)$ means the probability of event E, although $Prob(E)$, $\mathbb{P}(E)$ can also be used.

Sample Spaces

Sample space S: the set of all possible outcomes of an experiment.

Coin flip	{Heads, Tails}
Flipping two coins	$\{(H,H),(H,T),(T,H),(T,T)\}$
Roll of 6-sided die	{1,2,3,4,5,6}
Weather today	{Sunny, Rainy, Snowy, Windy}
Number of emails in a day	$\{z \mid z \in \mathbb{Z}, z \geq 0\}$
YouTube hours in a day	$\{z \mid z \in \mathbb{R}, 0 \leq z \leq 24\}$

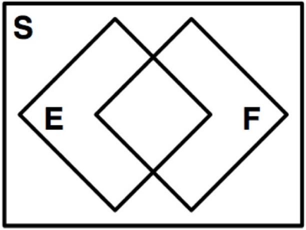
Events

Event E: a subset of sample space S, $E \subset S$. A set of possible outcomes when an experiment is performed

Coin comes up heads	{Heads}
One head and one tail on two flips	$\{(H, T), (T, H)\}$
Die roll is less than 3	{1,2}
Weather is wet	{Rainy,Snowy}
Number of emails is less than 20	$\{z \mid z \in \mathbb{Z}, 0 \leq z \leq 20\}$
Wasted day (at least 5 hours on YT)	$\{z \mid z \in \mathbb{R}, 5 \leq z \leq 24\}$

Set Operations on Events

Suppose E and F are events in sample space S i.e. $E, F \subset S$



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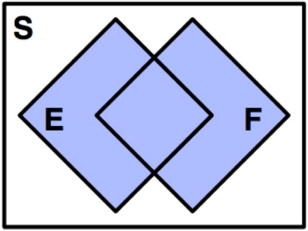

$E \cup F$ is the event consisting of all the outcomes in E or F. $\cup$ is called the <u>union</u>.

Set Operations on Events

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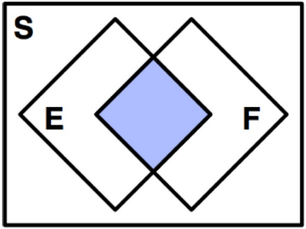
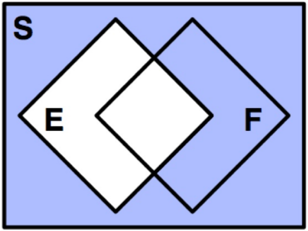


$E \cap F$ is the event consisting of all the outcomes in both E and F. $\cap$ is called the intersection.

Set Operations on Events

Suppose E and F are events in sample space S, $E, F \subset S$



E^c is the event consisting of all the outcomes not in E. ^{c} is called the complement.

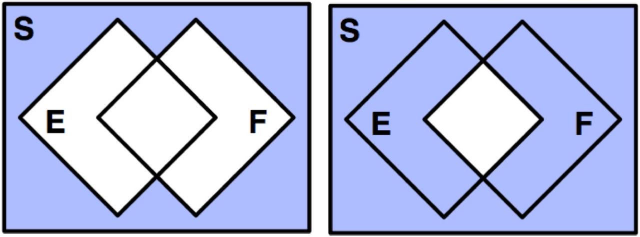
Set Operations on Events

Suppose E, F, G are sets. Basic properties of set union and intersection:

- $E \cup F = F \cup E$ and $E \cap F = F \cap E$
- $(E \cup F) \cup G = E \cup (F \cup G)$ and $(E \cap F) \cap G = E \cap (F \cap G)$
- $E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$ and
  $E \cup (F \cap G) = (E \cup F) \cap (E \cup G)$

Set Operations on Events

Suppose E and F are events in sample space S i..e $E, F \subset S$



$(E \cup F)^c = E^c \cap F^c \qquad (E \cap F)^c = E^c \cup F^c$

$(\bigcup_{i=1}^{n} E_i)^c = \bigcap_{i=1}^{n} E_i^c \qquad (\bigcap_{i=1}^{n} E_i)^c = \bigcup_{i=1}^{n} E_i^c$

(DeMorgan's Laws)

Axioms for Events

If E and F are events then so are:

- $E \cup F$
- $E \cap F$
- E^c and F^c

Consequently:

- For events E_i, $i = 1, 2, \cdots n$ then
  - $E_1 \cup E_2$ is an event
  - $(E_1 \cup E_2) \cup E_3 = E_1 \cup E_2 \cup E_3$ is an event
  - $\cup_{i=1}^{n} E_i$ is an event
  - $\cap_{i=1}^{n} E_i$ is an event
- S is an event since $S = E \cup E^c$ for any event E.
- The empty set $\emptyset$ is an event since $S^c = \emptyset$.
- Axioms really needed for sets with infinite number of elements (so n could be infinite). A technicality, but we'll confine ourselves to intersections, unions and complements when talking about events.

Axioms of Probability

Think of

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n}$$

where $n(E)$ is the number of times event E occurs in n trials (we'll come back to this later).
What basic properties does this quantity always have ?

Axioms of Probability

Axiom 1 $0 \leq P(E) \leq 1$

Axiom 2 $P(S) = 1$, where S is sample space (set of all possible outcomes)

Axiom 3 If E and F are mutually exclusive ($E \cap F = \emptyset$) then $P(E \cup F) = P(E) + P(F)$. More generally,

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i)$$

provided $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

Implications of Axioms

$P(E^c) = 1 - P(E)$

- Since $S = E \cup E^c$ and $E \cap E^c = \emptyset$ then $P(S) = 1 = P(E) + P(E^c)$

E.g. $S =$ {Sunny, Rainy, Snowy, Windy}

- If $E =$ {Sunny} then E^c = {Rainy, Snowy, Windy}
- $P(E) + P(E^c) = 1$ so $P(E) = 1 - P(E^c)$
- P({Sunny}) = 1 - P({Rainy, Snowy, Windy})

Note $P(S) + P(S^c) = P(S) + P(\emptyset) = 1$ and $P(S) = 1$, so $P(\emptyset) = 0$ i.e. emptyset is just a formality.

Implications of Axioms

$E \subset F$ implies that $P(E) \leq P(F)$

- Since $F = E \cup (E^c \cap F)$ and $E \cap E^c = \emptyset$ then $P(F) = P(E) + P(E^c \cap F))$
- $P(E^c \cap F) \geq 0$ so $P(E) = P(F) - P(E^c \cap F)) \leq P(F)$

E.g. $S =${Sunny, Rainy, Snowy, Windy}

- If $E =${Rainy} then $F=${Rainy, Snowy}
- $P(\{\text{Rainy}\}) \leq P(\{\text{Rainy, Snowy}\})$

Implications of Axioms

$P(E \cup F) = P(E) + P(F) - P(E \cap F)$

- $E \cup F = E \cup (E^c \cap F)$ and $E \cap (E^c \cap F) = \emptyset$ (mutually exclusive)
- $F = (E \cap F) \cup (E^c \cap F)$, also mutually exclusive
- So $P(E \cup F) = P(E) + P(E^c \cap F)$
- and $P(F) = P(E \cap F) + P(E^c \cap F)$ i.e. $P(E^c \cap F) = P(F) - P(E \cap F)$

E.g. $S =$ {Sunny, Rainy, Snowy, Windy}

- If $E =$ {Rainy} then $F=$ {Snowy}
- P({Rainy,Snowy}) $=$ P({Rainy}) $+$ P({Snowy}) $-$ P({Rainy} and {Snowy})

Equally Likely Outcomes

In some experiments all outcomes are equally likely. E.g. tossing a fair coin:

- $S = \{Heads, Tails\}$
- $P(\{Heads\}) = P(\{Tails\}) = p$ (coin is fair).
- Using axioms:
  - $P(S) = P(\{Heads, Tails\}) = 1$
  - $P(\{Heads, Tails\}) = P(\{Heads\}) + P(\{Tails\}) = 2p = 1$. Solve to get $p = \frac{1}{2}$

Equally Likely Outcomes

Another example:

- $S = \{1, 2, \cdots, N\}$ and $P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$.
- Then $P(\{1\}) = \frac{1}{N}$, $P(\{2\}) = \frac{1}{N}$ etc

And for events consisting of multiple outcomes:

- $P(E) = \frac{\text{Number of outcomes in E}}{\text{Number of outcomes in S}} = \frac{|E|}{|S|}$
- E.g. $S = \{1, 2, 3, 4, 5, 6\}$ and $E = \{3, 4\}$ then $P(E) = \frac{2}{6}$.

Rolling Two Dice

Roll two 6-sided dice

- What is the probability that the dice sum to 7 ?

And for events consisting of multiple outcomes:

- Sample space $S = \{(1,1),(1,2),(1,3),\cdots,(6,5),(6,6)\}$
- Event $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
- $P(E) = \frac{6}{36} = \frac{1}{6}$

Tossing a Coin

Toss a fair coin twice:

- What is the probability that get two heads ?
- Sample space $S = \{(H, H), (H, T), (T, H), (T, T)\}$
- Event $E = \{(H, H)\}$
- $P(E) = \frac{1}{4}$
- What is the probability that get heads then tails ?
- Event $E = \{(H, T)\}$
- $P(E) = \frac{1}{4}$
- What is the probability that get one head and one tail?
- Event $E = \{(H, T), (T, H)\}$
- $P(E) = \frac{2}{4} = 0.5$

Drawing Balls from a Bag

Have an bag containing 4 red balls and 3 white balls. Draw 3 balls.

- What is $P(1 \text{ red ball and } 2 \text{ white balls drawn})$?
- Can draw 3 balls out of bag containing 7 balls in $\binom{7}{3} = 35$ ways. So sample space S is of size $|S| = 35$.
- Event E is of size $\binom{4}{1}\binom{3}{2} = 12$
- $P(1 \text{ red ball and } 2 \text{ white balls drawn}) = \frac{12}{35}$
- What is $P(\text{at least } 2 \text{ red balls drawn})$?
- Event E is of size $\binom{4}{2}\binom{3}{1} + \binom{4}{3} = 22$, $P(\geq 2 \text{ red balls drawn}) = \frac{22}{35}$
- What is $P(\text{at least } 2 \text{ white balls drawn})$?
- Event E is of size $\binom{3}{2}\binom{4}{1} + \binom{3}{3} = 13$, $P(\geq 2 \text{ white balls drawn}) = \frac{13}{35}$

Important Trick

Often its hard to count the number of times an event E occurs, but easy to count the number of time event E does not occur. Use $P(E) = 1 - P(E^c)$, where E^c is the event that E does not occur.

- We flip a coin 3 times. What is the probabilty that there is at least one heads ?
  - Sample space $|S| = 2^3 = 8$.
  - Event that no heads is $E^c = \{(T, T, T)\}$. $|E^c| = 1$ so $P(E^c) = \frac{1}{8}$.
  - Therefore $P(E) = 1 - P(E^c) = 1 - \frac{1}{8}$ is the probability of one or more heads.
  - What if we flipped the coin 10 times ? 100 times ?
- We toss a dice twice. What is the probability that the sum is greater than 3 ?
  - Sample space $|S| = 6^2 = 36$.
  - Event that less than or equal to three is $E^c = \{(1,1),(1,2),(2,1)\}$. $|E^c| = 3$ so $P(E^c) = \frac{3}{36}$.
  - Therefore $P(E) = 1 - P(E^c) = 1 - \frac{3}{36}$ is the probability the sum is greater than 3.

Birthdays

What is the probability of event E that of n people two or more share the same birthday (regardless of year) ?

Birthdays

What is the probability event E that of n people one or more of them shares a birthday with <u>you</u> ?
Let's ask the complement E^c: of n people what is the probability that none of them shares a birthday with <u>you</u> ?

- $|S| = 365^n$
- $|E^c| = 364^n$
- $P(E^c) = \frac{364^n}{365^n}$
- $P(E) = 1 - P(E^c)$.

Some values:

- When $n = 23$ then $P(\text{no matching birthdays}) \approx 0.94$
- When $n = 75$ then $P(\text{no matching birthdays}) \approx 0.81$
- When $n = 100$ then $P(\text{no matching birthdays}) \approx 0.76$

Why are these probabilities so much higher than before ?

Poker Hands

- Straight flush is 5 consecutive cards of same suit.
- What is P(straight flush) ?
- Sample space $|S| = \binom{52}{5} = 2598960$
- 4 suits. For each suit (each with 13 cards) can get a straight flush 10 different ways. Event $|E| = 10 \times 4$.
- $P(\text{straight flush}) = \frac{40}{2598960} \approx 1.5 \times 10^{-5}$
- What is P(four of a kind) ?
- 13 ways to select 4 cards of the same kind. 5th card can be selected from remaining 12 kinds, and from each of 4 suits i.e. 12×4 ways. Event $|E| = 13 \times 12 \times 4 = 624$.
- $P(\text{four of a kind}) = \frac{624}{2598960} \approx 12.4 \times 10^{-4}$