Overview

- Cumulative Distribution Functions Continued
- Conditional Probability Density Function
- Chain Rule for PDFs
- Bayes Rule for PDFs
- Independence

Cumulative Distribution Functions Continued

Suppose X and Y are two random variables.

- $F_{XY}(x, y) = P(X \leq x \text{ and } Y \leq y)$ is the cumulative distribution function for X and Y
- When X and Y are independent then:

$$P(X \leq x \text{ and } Y \leq y) = P(X \leq x)P(Y \leq y)$$

$$i.e. \qquad F_{XY}(x, y) = F_X(x)F_Y(y)$$

Cumulative Distribution Functions Continued

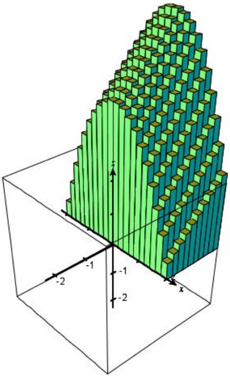
When X and Y are discrete random variables taking values $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$:

- $F_{XY}(x, y) = \sum_{i:x_i \leq x} \sum_{j:y_j \leq y} P(X = x_i \text{ and } Y = y_j)$

When X and Y are jointly continuous-valued random variables there exists a probability density function (PDF) $f_{XY}(x, y) \geq 0$ such that:

- $F_{XY}(x, y) = \int_{\infty}^{x} \int_{-\infty}^{y} f_{XY}(u, v) du\ dv$

Can think of $P(u \leq X \leq u + du \text{ and } v \leq Y \leq v + dv) \approx f_{XY}(u, v) du\ dv$ when du, dv are infinitesimally small.



Conditional Probability Density Function

Suppose X and Y are two continuous random variables with joint PDF $f_{XY}(x, y)$. Define conditional PDF:

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Compare with conditional probability for discrete RVs:

$$P(X = x|Y = y) = \frac{P(X = x \text{ and} Y = y)}{P(Y = y)}$$

Chain Rule for PDFs

Since

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

the chain rule also holds for PDFs:

$$f_{XY}(x,y) = f_{X|Y}(x|y)f_Y(y) = f_{Y|X}(y|x)f_X(x)$$

Also,

- $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \frac{\int_{-\infty}^{\infty} f_{XY}(x,y)dx}{f_Y(y)} = \frac{f_Y(y)}{f_Y(y)} = 1$
- We can marginalise PDFs:

$$\begin{aligned}\int_{-\infty}^{\infty} f_{XY}(x,y)dy &= \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x)dy \\ &= f_X(x)\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = f_X(x)\end{aligned}$$

Bayes Rule for PDFs

Since

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

then

$$f_{X|Y}(x|y)f_Y(y) = f_{XY}(x,y) = f_{Y|X}(y|x)f_X(x)$$

and so we have Bayes Rule for PDFs:

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f_Y(y)}{f_X(x)}$$

Independence

Suppose X and Y are two continuous random variables with joint PDF $f_{XY}(x,y)$. Then X are Y are independent when:

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

Why ?

$$\begin{aligned} P(X \leq x \text{ and } Y \leq y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v) du dv \\ &= \int_{-\infty}^{x} f_X(u) du \int_{-\infty}^{y} f_Y(v) dv \\ &= P(X \leq x) P(Y \leq y) \end{aligned}$$

Example

Suppose random variable $Y = X + M$, where $M \sim N(0,1)$. Conditioned on $X = x$, what is the PDF of Y ?

- $f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{(y-x)^2}{2})$

Suppose that $X \sim N(0,\sigma)$. What is $f_{X|Y}(x|Y)$?

- Use Bayes Rule:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} \\ &= \frac{\frac{1}{\sqrt{2\pi}} \exp(-\frac{(y-x)^2}{2}) \times \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})}{f_Y(y)} \end{aligned}$$

- $f_Y(y)$ is just a normalising constant (so that the area under $f_{X|Y}(x|y)$ is 1).