ST3009 Mid-Term Test 2017

Attempt **all** questions. Time: 1 hour 30 mins.

1. Consider two random variables X and Y that take values in set {1,2, … ,n}.

a) Give the definition of the conditional probability P(Y=y|X=x)

[5 points]

Solution: P(Y=y|X=x) = P(Y=y and X=x)/P(X=x)

b) Prove the marginalization property that $P(X=x)=\sum_{i=1}^{n} P(X=x \; and \; Y=i)$

[10 points]

Solution:

$\sum_{i=1}^{n} P(X=x \; and \; Y=i)=\sum_{i=1}^{n} P(Y=i|X=x)P(X=x)=P(X=x)$

c) Using the marginalization property in (b) prove that $\sum_{i=1}^{n} P(Y=i|X=x)=1$

[10 points]

Solution: $\sum_{i=1}^{n} P(Y=i|X=x)=\frac{\sum_{i=1}^{n} P(Y=i \; and \; X=x)}{P(X=x)}=\frac{P(X=x)}{P(X=x)}$=1 where we have used the fact that

$\sum_{i=1}^{n} P(\, Y=i \; and \; X=x)=P(X=x)$

d) A bag of insects contains 10 crickets and 5 spiders. I draw two insects from the bag, without replacement. Given that the second insect was a spider, what is the probability that the first insect drawn was also a spider ? Hint: use marginalisation to calculate the probability that the second insect was a spider.

[5 points]

Solution: Let RV X_1=1 if draw a spider first and 0 otherwise, and X_2=1 if draw a spider second and 0 otherwise. We want

P(X_1=1|X_2=1)=P(X_1=1 and X_2=1)/P(X_2=1)=0.2856

Now P(X_1=1 and X_2=1) = 5/15 x 4/14 = 0.0952. Also, P(X_2=1) = P(X_2=1 and X_1=1) +P(X_2=1 and X_1=0) = 5/15 x 4/14 + 10/15 x 5/14 = 0.333.

2.

a) State Bayes Rule.

[5 points]

Solution: For events E and F: P(E|F) = P(F|E) P(E)/P(F)

b) Suppose 48% of people in the population support presidential candidate T and the rest support candidate C. When asked which candidate they support, 75% of supporters of candidate T answer truthfully, the others falsely answering that they support C. When asked, 100% of supporters of C answer truthfully. Suppose a person answers that they support candidate C. What is the probability that in fact they support candidate T.

[10 points]

Solution: Let E be the event that the person supports T, and F be the event that they say they support C. We want P(E|F) = P(F|E)P(E)/P(F). Now P(F|E) = 0.25, P(E)=0.48 and P(F)=P(F|E)P(E)+P(F|E^c)P(E^c)= 0.25 x 0.48 + 1 x (1-0.48) = 0.64. So P(E|F) = 0.25 x 0.48 / 0.64 = 0.1875.

3. Five people play the game of "odd-man-out" to determine who pays for a meal. In this game, each person flips a coin. If one person's coin comes up different to all others (i.e. there is one H and four T's or there is one T and four H's), then that person pays. Otherwise, everyone flips again. They go on doing this until someone is chosen.

a) What is the probability in each round that one person's coin comes up different to all others ?

[5 points]

Solution: The sample space of five coin flips has 2^5=32 outcomes. Ten of these end the game (five are where person comes up heads and rest tails, five where they come up tails and the rest heads), so the probability that the game ends at a given round is $10/2^5$=0.3125 and the probability that it continues is $1-10/2^5$.

b) Define the expectation of a random variable.

[5 points]

Solution: Suppose RV X can take values $x_1, x_2, \ldots, x_n$. Then the expectation is $E[X] = \sum_{i=1}^{n} x_i P(X = x_i)$.

c) What is the expected number of times they must flip before they know who should pay? Hint $\sum_{i=1}^{\infty} ix^{i-1} = \frac{1}{(1-x)^2}$

[10 points]

Solution: Let RV X be the number of rounds in the game. Then E[X]= $1\text{x}10/2^5+2(1-10/2^5)10/2^5+3\text{x}(1-10/2^5)^2 10/2^5+\ldots$ That is, E[X]= $10/2^5(1+2(1-10/2^5)+3(1-10/2^5)^2+\ldots)$ or $E[X] = (1-x)\sum_{i=1}^{\infty} ix^{i-1} = \frac{1-x}{(1-x)^2} = \frac{1}{1-x}$ with $x=(1-10/2^5)$.

b) Write a short matlab simulation of this game that outputs the number of flips made.

[10 points]

Example solution:

```
p=10/2^5;
count = 1;
while (rand>p), count=count+1; end
disp(count)
```

Another example:

```
C=(rand(1,5)>0.5); % five coin tosses
count=1;
while (sum(C)==1 || sum(C)==4),
        C=(rand(1,5)>0.5);
        count=count+1;
end
disp(count)
```