ST3009 Mid-Term Test 2016

Attempt **all** questions. Time: 1 hour 30 mins.

1. Suppose we roll a red die and a green die.

(i) What is the sample space for this experiment?

Solution: S={(1,1),(1,2),…,(1,6),(2,1),(2,2),…,(2,6),…,(6,1),(6,2),…,(6,6)}

(ii) What is the probability that the number on the green die is larger than the number on the red die?

Solution: This corresponds to the event E={(1,2),…,(1,6),(2,3),..,(2,6),(3,4),..,(3,6),(4,5),(4,6),(5,6)}.
This set has 5+4+3+2+1=15 elements and the sample space has 36 elements, so the probability of the event E is 15/36.

(iii) Define what it means for two events E and F to be independent.

Solution: $P(E \cap F) = P(E)P(F)$.

(iv) Let event E be that the sum equals 2 or 3 and event F be that the sum equals 3. Are E and F independent? Explain with reference to the definition given above.

Solution: Event E={(1,1),(1,2),(2,1)}. Event F={(1,2),(2,1)}.
P(E)=3/36=1/6, P(F)=2/36=1/18, $P(E \cap F)$=P({(1,2),(2,1)}) =2/36=1/18.
P(E)P(F)=1/6x1/18 is not equal to $P(E \cap F)$=1/18, so the events are not independent.

2.

(i) State Bayes Rule.
Solution: For events E and F: P(E|F) = P(F|E)P(E)/P(F)

(ii) Suppose 1% of computers are infected with a virus. There is an imperfect test for detecting the virus. When applied to a computer with the virus the test gives a positive result 90% of the time. When applied to a computer which does not have the virus, the test gives a negative result 99% of the time. Suppose that the test is positive for a computer. What is the probability that the computer has the virus ?
Solution: Let E be the event that a computer has the virus and F the event that the test is positive. P(E)=0.01, P(F|E)=0.9, $P(F)=P(F|E)P(E)+P(F|E^c)P(E^c)$=0.9x0.01+(1-0.99)x(1-0.01)=0.0189. So by Bayes Rule P(E|F)=0.9x0.01/0.0189=0.476.

3. You invent a game where the player bets €1, and rolls two dice. If the sum is 7, the player wins €k, and otherwise loses their bet.

(i) Define the expectation and variance of a discrete random variable.

Solution: For random variable X taking values $x_1, \ldots, x_n$ the expected value is $E[X]=x_1P(X=x_1)+\ldots x_nP(X=x_n)$. The variance is $Var(X)=E[(X-E[X])^2] = (x_1-E[X])^2P(X=x_1)+\ldots (x_n-E[X])^2P(X=x_n)$.

(ii) What is the expected reward in this game ?
Solution: Sample space S for the two dice is of size 36. Event that dice sum to 7 is E={(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}. So the probability P(E)=6/36=1/6. When event E occurs player wins k euros. Otherwise they lose 1 euro. Let X be a random variable with equals k on event E and -1 otherwise. E[X]= kxP(E)-1x(1-P(E)) = k/6-5/6.

(iii) What value of k makes the game fair (i.e. makes the expected reward zero) ? What is the variance of the reward in this case ?
Solution: We want to find k such that E[X]=k/6-5/6=0. Choose k=5.
$Var(X)=(5-0)^2x1/6+(-1-0)^2x5/6=5^2/6+5/6=5$.

(iv) For two independent random variables X and Y show that Var(X+Y)=Var(X)+Var(Y). Hint: Recall that E[X+Y]=E[X]+E[Y] and that when X and Y are independent then E[XY]=E[X]E[Y]
Solution: See notes.

(v) Suppose that you play the game 2 times in a row with k=5. What is the variance of the reward now (i.e. of the aggregate winnings after playing 2 times)? What is the variance after 100 plays ?
Solution: Let X be the reward the first time the game is played and Y the reward the second time it is played. E[X+Y]=E[X]+E[Y]=0+0=0. Var(X+Y)=Var(X)+Var(Y)=5+5=10. The expectation after 100 plays is still 0 and the variance is 100x5=500.