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Extra Questions ST3009: Statistical Methods for Computer Science

Question 1. Consider the following game: first a coin with $P(heads) = q$ is tossed once. If the coin comes up tails, then you roll a 4-sided die; otherwise, you roll a 6-sided die. You win the amount of money (in euros) corresponding to the given die roll. Let X be an indicator random variable for the coin toss ($X = 0$ if toss is tails; $X = 1$ if toss is heads), and let Y be the random variable corresponding to the amount of money that you win.

(a) Compute the joint PMF $P(X = x \text{ and } Y = y)$

(b) Compute the conditional PMF $P(X = x|Y = y)$, again as a function of q. Supposing that it is known that (on some trial of this game) you made €2 or less, determine the probability that the initial coin toss was heads, as a function of q.

(c) Assume that you have to pay €3 each time that you play this game. Determine, as a function of q, how much money you will win or lose on average. For what value of q do you break even?

Solution:

(a) We have

$$P(X = x \text{ and } Y = y) = \begin{cases} \frac{q}{6} & \text{if } x = 1 \text{ and } y \in \{1,2,3,4,5,6\} \text{ (with prob } q \text{ roll 6-sided die)} \\ \frac{(1-q)}{4} & \text{if } x = 0 \text{ and } y \in \{1,2,3,4\} \text{ (with prob } 1-q \text{ roll 4-sided die)} \\ 0 & \text{otherwise} \end{cases}$$

(b) By marginalising $P(X = x \text{ and } Y = y)$ we have

$$P(Y = y) = \begin{cases} \frac{(1-q)}{4} + \frac{q}{6} & \text{if } y \in \{1,2,3,4\} \\ \frac{q}{6} & \text{if } y \in \{5,6\} \\ 0 & \text{otherwise} \end{cases}$$

and so

$$P(X = x|Y = y) = \frac{P(X = x \text{ and } Y = y)}{P(Y = y)}$$
$$= \begin{cases} \frac{3(1-q)}{3-q} & \text{if } x = 0 \text{ and } y \in \{1,2,3,4\} \\ \frac{2q}{3-q} & \text{if } x = 1 \text{ and } y \in \{1,2,3,4\} \\ 1 & \text{if } x = 1 \text{ and } y \in \{5,6\} \\ 0 & \text{otherwise} \end{cases}$$

(c) Given the set-up of the game, the expected amount that we win is given $E[Y]$. We compute

$$E[Y] = \sum_{y=1}^{6} yP(Y = y) = q + \frac{5}{2}$$

and so we break even when $q + \frac{5}{2} \geq 3$ i.e. when $q \geq \frac{1}{2}$.

Question 2. An edge detector is applied in order to detect edges in an image. Conditioned on an edge being present at some position, the detector response is Gaussian with mean 0 and variance σ^2, whereas conditioned on no edge being present, the detector response is zero-mean Gaussian with variance 1. Any position in the image has a probability p of containing an edge.

(a) Compute the mean and variance of the detector response X

(b) Compute the conditional probability of an edge being present given that $|X| \geq 10$. Your answer should be expressed in terms of p, σ and the Gaussian CDF $Prob(Z \leq z) = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$

Solution:

(a) We condition on the presence/absence of the edge, an event denoted by G. We have:

$$E[X] = pE[X|G] + (1-p)E[X|G^c] = 0$$
$$var(X) = E[X^2] - E[X]^2 = E[X^2] = pE[X^2|G] + (1-p)E[X^2|G^c] = p\sigma^2 + (1-p)$$

(b) By Bayes rule, we have:

$$\begin{aligned} P(G|\ |X| \geq 10) &= \frac{P(|X| \geq 10|G)P(G)}{P(|X| \geq 10)} \\ &= \frac{P(|X| \geq 10|G)p}{pP(|X| \geq 10|G) + (1-p)P(|X| \geq 10|G^c)} \\ &= \frac{P(|Z| \geq 10/\sigma|G)p}{pP(|Z| \geq 10/\sigma) + (1-p)P(|Z| \geq 1)} \end{aligned}$$

where $Z \sim N(0,1)$ is a Normally distributed random variable with mean 0 and variance 1. Hence,

$$P(G|\ |X| \geq 10) = \frac{2\Phi(-10/\sigma)p}{2\Phi(-10/\sigma)p + 2\Phi(-10)(1-p)}$$

where $P(|Z| \geq 10/\sigma^2) = 1 - (\Phi(10/\sigma) - \Phi(-10/\sigma)) = 2\Phi(-10/\sigma)$ by the symmetry of the Gaussian distribution.

Question 3. I am playing in a racquetball tournament, and I am up against a player I have watched but never played before. I consider three possibilities for my prior model: we are equally talented, and each of us is equally likely to win each game; I am slightly better, and therefore I win each game independently with probability 0.6; he is slightly better, and thus he wins each game independently with probability 0.6. Before we play, I think that each of these three possibilities is equally likely. In our match we play until one player wins three games. I win the second game, but he wins the first, third, and fourth. After this match, in my posterior modeL with what probability should I believe that my opponent is slightly better than I am'?

Solution: Use Bayes Rule. Let E be the observed set of wins and let F be the event that the opponent is better than me. By Bayes,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Now $P(F) = \frac{1}{3}$ since my prior is that all three possibilties are equally likely, $P(E|F) = 0.6 \times (1 - 0.6) \times 0.6 \times 0.6$ and

$$\begin{aligned} P(E) &= P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H) \\ &= 0.6^3(1-0.6)\frac{1}{3} + 0.5^3(1-0.5)\frac{1}{3} + (1-0.6)^3 0.6\frac{1}{3} \end{aligned}$$

where G is the event that we are equally talented and H is the event that I am slightly better. Plug these values into Bayes rule to obtain the answer.

Question 4. The coupon collectors problem is as follows. Suppose that each box of cereal contains one of n different coupons. Once you obtain one of every type of coupon, you can send in for a prize. Assume that the coupon in each box is chosen independently and uniformly at random from the n possibilities and that you do not collaborate with others to collect coupons. Let X be the number of boxes bought until at least one of every type of coupon is obtained.

(a) Give an expression for the expected value of X ? Hint: work in terms of X_i, the number of boxes bought while you have exactly $i-1$ coupons, and note that $\sum_{j=1}^{\infty} j(1-p)^j p = \frac{1}{p}$.

(b) Use Markov's inequality to give an upper bound on the probability that X is greater than $10n$.

Solution:

(a) Let X_i be the number of boxes bought while you have exactly $i-1$ coupons. Then $X = \sum_{i=1}^{n} X_i$. When exactly $i-1$ coupons have been found, the probability of obtaining a new coupon is

$$p_i = 1 - \frac{i-1}{n}$$

and so

$$E[X_i] = \sum_{j=1}^{\infty} j(1-p_i)^j p_i = \frac{1}{p_i} = \frac{n}{n-i+1}$$

Using the linearity of expectations, $E[X] = \sum_{i=1}^{n} E[X_i]$ and so

$$E[X] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{i=1}^{n} \frac{1}{i}$$

(b) Markov's inequality is

$$P(X \geq 10n) \leq \frac{E[X]}{10n} = \frac{1}{10} \sum_{i=1}^{n} \frac{1}{i}$$

Question 5. Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive-or of the ith pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$ be the number of that equal 1.

(a) Show that each Y_i is 0 with probability 0.5

(b) Show that the Y_i are not mutually independent

(c) Show that the Y_i satisfy the property $E[Y_iY_j] = E[Y_i]E[Y_j]$

Solution:

(a) We pick two bits. Let Z_1 be the value of the first bit and Z_2 the value of the second bit. The first bit $Z_1 = 1$ with probability 0.5 and 0 with probability 0.5. Similarly the second bit Z_2. Both bits are independent. So $P(Y_i = 0) = P(Z_1 = Z_2) = P(Z_1 = 0 \text{ and } Z_2 = 0 \text{ or } Z_1 = 1 \text{ and } Z_2 = 1) = 0.5^2 + (1 - 0.5)^2 = 0.25 + 0.25 = 0.5$

(b) Suppose pair j has the same first bit as pair i, then they will not be independent. Formally, for independence we require $P(Y_i = 0 \text{ and } Y_j = 0) = P(Y_i = 0)P(Y_j = 0) = 0.5 \times 0.5 = 0.25$ for all pairs of bits i and j. Let $Z_{1,i}$ be the value of the first bit in pair i, $Z_{2,i}$ be the value of the seond bit in pair i. Similarly $Z_{1,j}$ and $Z_{2,j}$ for pair j. Suppose $Z_{1,i} = Z_{1,j}$ i.e. the first bit is in fact the same for both pairs. Then

$$\begin{aligned} P(Z_{1,i} = Z_{2,i}) &= 0.5^2 + (1 - 0.5)^2 = 0.5 \\ P(Z_{1,j} = Z_{2,j} | Z_{1,i} = Z_{2,i}) &= P(Z_{1,i} = Z_{2,i} = Z_{2,j}) = 0.5^3 + (1 - 0.5)^3 = 0.25 \end{aligned}$$

and so

$$P(Y_i = 0 \text{ and } Y_j = 0) = P(Y_j = 0 | Y_i = 0)P(Y_i = 0) = 0.25 \times 0.5 = 0.125 \neq 0.25$$

(c) $E[Y_iY_j] = 1 \times P(Y_i = 1 \text{ and } Y_j = 1)$ and $E[Y_i] = 1 \times P(Y_i = 1)$, so we need to show that $P(Y_i = 1 \text{ and } Y_j = 1) = P(Y_i = 1)P(Y_j = 1)$. By definition holds when Y_i and Y_j are independent i.e. when pairs i and j share no bits in common. When pairs i and j share one bit in common (they cannot share two bits, as then they would be the same pair), say the first bit $Z_{1,i} = Z_{1,j}$, then

$$\begin{aligned} P(Y_i = 1 \text{ and } Y_j = 1) &= P(Z_{1,j} \neq Z_{2,j} \text{ and } Z_{1,i} \neq Z_{2,i}) \\ &= P(Z_{2,j} \neq Z_{1,i} \text{ and } Z_{2,i} \neq Z_{1,i}) \end{aligned}$$

But $Z_{2,j}$ and $Z_{2,i}$ are independent, so

$$\begin{aligned} P(Z_{2,j} \neq Z_{1,i} \text{ and } Z_{2,i} \neq Z_{1,i}) &= P(Z_{2,j} \neq Z_{1,i})P(Z_{2,i} \neq Z_{1,i}) \\ &= P(Z_{2,j} \neq Z_{1,i})P(Y_i = 1) \end{aligned}$$

Now $P(Z_{2,j} \neq Z_{1,i}) = 0.5(1-0.5)+(1-0.5)0.5 = 0.5 = P(Y_j = 1)$ since $Z_{2,j}$ and $Z_{1,i}$ are independent, and so $P(Y_i = 1 \text{ and } Y_j = 1) = P(Y_i = 1)P(Y_j = 1)$ as required.