TRINITY COLLGE DUBLIN

School of Computer Science and Statistics

Extra Questions ST3009: Statistical Methods for Computer Science

NOTE: There are more example questions in Chapter 4 of the course textbook "A First Course in Probability" by Sheldon Ross.

Question 1. Consider an experiment where we toss 3 fair coins. Let random variable Y be the number of heads that appear. The sample space is $\{(h,h,h),(t,h,h),(h,t,h),\ldots\}$ and recall that a random event is a subset of the sample space. What random event corresponds to $Y = 1$? What event corresponds to $Y = 2$? What event corresponds to $Y = 3$? What are the probabilities of these three events ?

Solution

- $\{(h,t,t),(t,h,t),(t,t,h)\}$
- $\{(h,h,t),(h,t,h),(t,h,h)\}$
- $\{(h,h,h)\}$
- The sample space has $2^3 = 8$ possible events. $P(X = 1) = 3/8$, $P(X = 2) = 3/8$, $P(X = 3) = 1/8$.

Question 2. Two balls are randomly drawn from an urn containing 3 white, 3 red, and 5 black balls. Suppose that we win €1 for each white ball selected and lose €1 for each red ball selected. Let X denote our total winnings from the experiment. What is the random event corresponding to $X = 0$? What is the event corresponding to $X = 2$? What are the probabilities of these events ?

Solution

- $\{(b,b),(r,w),(w,r)\}$,
- $\{(r,r)\}$
- There are $\binom{5}{2}$ ways to draw two black balls, $\binom{3}{1}\binom{3}{1}$ ways to choose a red and white. So $P(X = 0) = (\binom{5}{2} + \binom{3}{1}\binom{3}{1})/\binom{11}{2} = 0.3455$. Alternatively, the probability of drawing two black balls is $5/11 \times 4/10$, the probability of drawing a red then a white is $3/11 \times 3/10$ and of a white then red is $3/11 \times 3/10$ giving an overall probability $P(X = 0) = 5/11 \times 4/10 + 2 \times 3/11 \times 3/10 = 0.3455$.
- $P(X = 2) = 3/11 \times 2/10$.

Question 3. Suppose the cumulative distribution function of random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \le x < 1 \\ 3/5 & 1 \le x < 2 \\ 4/5 & 2 \le x < 3 \\ 9/10 & 3 \le x < 3.5 \\ 1 & b \ge 3.5 \end{cases}$$

Calculate the probability mass function of X.

Solution The value of $F(x)$ changes only at $x = 0, 1, 2, 3, 3.5$ and so these are the possible values of discrete-valued random variable X. We have $P(X = 0) = F(0) = 1/2$, $P(X = 1) = F(1) - F(0) = 3/5 - 1/2$, $P(X = 2) = F(2) - F(1) = 4/5 - 3/5$, $P(X = 3) = F(3) - F(2) = 9/10 - 4/5$, $P(X = 3/5) = F(3.5) - F(3) = 1 - 9/10$.

Question 4. Three balls are drawn independently with replacement from bag contains 3 white and 2 red balls. Let X be the number of red balls drawn. Calculate the PMF and CDF of X

Solution

- PMF. X takes values 0,1,2. $P(X = 0) = Prob(3 \text{ white balls}) = (3/5)^3$. $P(X = 1) = Prob(1 \text{ red ball}) = \binom{3}{1}(2/5)(3/5)^2$, $P(X = 1) = Prob(2 \text{ red balls}) = \binom{3}{2}(2/5)^2(3/5)$.
- CDF.

$$F(x) = \begin{cases} 0 & x < 0 \\ (3/5)^3 & 0 \le 0 < 1 \\ \binom{3}{1}(2/5)(3/5)^2 & 1 \le x < 2 \\ \binom{3}{2}(2/5)^2(3/5) & 2 \le x \end{cases}$$

Question 5. Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Solution If we let X equal the number of heads (successes) that appear, then X is a binomial random variable with parameters $n = 5$, $p = 0.5$ and so

$$P(X = 0) = \binom{5}{0} 0.5^0 (1 - 0.5)^5$$
$$P(X = 1) = \binom{5}{1} 0.5^1 (1 - 0.5)^4$$
$$P(X = 2) = \binom{5}{2} 0.5^2 (1 - 0.5)^3$$
$$P(X = 3) = \binom{5}{3} 0.5^3 (1 - 0.5)^2$$
$$P(X = 4) = \binom{5}{4} 0.5^4 (1 - 0.5)^1$$
$$P(X = 5) = \binom{5}{5} 0.5^5 (1 - 0.5)^0$$

Question 6. Suppose that the probability that an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain at most 1 defective item.

Solution $\binom{10}{0}(0.1)^0(0.9)^10 + \binom{10}{1}(0.1)^1(0.9)^9 = 0.7361$.

Question 7. It is known that screws produced by a certain company will be defective with probability 0.01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Solution If X is the number of defective screws in a package, then X is a binomial random variable with parameters $(10, 0.01)$. Hence, the probability that a package will have to be replaced is $1 - P(X = 0) - P(X = 1) = 1 - \binom{10}{0} 0.01^0 (1 - 0.01)^{10} - \binom{10}{1} 0.01^1 (1 - 0.01)^9 \approx 0.004$

Question 8. Suppose that a particular trait (such as eye color or left-handedness) of a person is classified on the basis of one pair of genes, and suppose also that d represents a dominant gene and r a recessive gene. Thus, a person with dd genes is purely dominant, one with rr is purely recessive, and one with rd is hybrid. The purely dominant and the hybrid individuals are alike in appearance. Children receive 1 gene from each parent. If, with respect to a particular trait, 2 hybrid parents have a total of 4 children, what is the probability that 3 of the 4 children have the outward appearance of the dominant gene?

Solution If we assume that each child is equally likely to inherit either of 2 genes from each parent, the probabilities that the child of 2 hybrid parents will have dd, rr, and rd pairs of genes are, respectively, 1/4, 1/4, 1/2. Hence, since an offspring will have the outward appearance of the dominant gene if its gene pair is either dd or rd, it follows that the number of such children is binomially distributed with parameters (4, 3/4). Thus, the desired probability is $\binom{4}{3}(3/4)^3(1/4)^1 = 27/64$.