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Extra Questions ST3009: Statistical Methods for Computer Science

Question 1. Consider the following game: first a coin with $P(heads) = q$ is tossed once. If the coin comes up tails, then you roll a 4-sided die; otherwise, you roll a 6-sided die. You win the amount of money (in euros) corresponding to the given die roll. Let X be an indicator random variable for the coin toss ($X = 0$ if toss is tails; $X = 1$ if toss is heads), and let Y be the random variable corresponding to the amount of money that you win.

(a) Compute the joint PMF $P(X = x \text{ and } Y = y)$

(b) Compute the conditional PMF $P(X = x|Y = y)$, again as a function of q. Supposing that it is known that (on some trial of this game) you made €2 or less, determine the probability that the initial coin toss was heads, as a function of q.

(c) Assume that you have to pay €3 each time that you play this game. Determine, as a function of q, how much money you will win or lose on average. For what value of q do you break even?

Question 2. An edge detector is applied in order to detect edges in an image. Conditioned on an edge being present at some position, the detector response is Gaussian with mean 0 and variance σ^2, whereas conditioned on no edge being present, the detector response is zero-mean Gaussian with variance 1. Any position in the image has a probability p of containing an edge.

(a) Compute the mean and variance of the detector response X

(b) Compute the conditional probability of an edge being present given that $|X| \geq 10$. Your answer should be expressed in terms of p, σ and the Gaussian CDF $Prob(Z \leq z) = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$

Question 3. I am playing in a racquetball tournament, and I am up against a player I have watched but never played before. I consider three possibilities for my prior model: we are equally talented, and each of us is equally likely to win each game; I am slightly better, and therefore I win each game independently with probability 0.6; he is slightly better, and thus he wins each game independently with probability 0.6. Before we play, I think that each of these three possibilities is equally likely. In our match we play until one player wins three games. I win the second game, but he wins the first, third, and fourth. After this match, in my posterior modeL with what probability should I believe that my opponent is slightly better than I am'?

Question 4. The coupon collectors problem is as follows. Suppose that each box of cereal contains one of n different coupons. Once you obtain one of every type of coupon, you can send in for a prize. Assume that the coupon in each box is chosen independently and uniformly at random from the n possibilities and that you do not collaborate with others to collect coupons. Let X be the number of boxes bought until at least one of every type of coupon is obtained.

(a) Give an expression for the expected value of X ? Hint: work in terms of X_i, the number of boxes bought while you have exactly $i - 1$ coupons, and note that $\sum_{j=1}^{\infty} j(1-p)^j p = \frac{1}{p}$.

(b) Use Markov's inequality to give an upper bound on the probability that X is greater than $10n$.

Question 5. Suppose that we flip a fair coin n times to obtain n random bits. Consider all $m = \binom{n}{2}$ pairs of these bits in some order. Let Y_i be the exclusive-or of the ith pair of bits, and let $Y = \sum_{i=1}^{m} Y_i$ be the number of that equal 1.

(a) Show that each Y_i is 0 with probability 0.5

(b) Show that the Y_i are not mutually independent

(c) Show that the Y_i satisfy the property $E[Y_i Y_j] = E[Y_i]E[Y_j]$