ST3009 Mock Mid-Term Test

Attempt **all** questions. Time: 1 hour 30 mins.

1. (i) Define the terms "sample space", "event" and "random variable" and give an example of each. [10 points]

   (ii) What is an indicator random variable and what is the probability mass function of a discrete random variable ? [5 points]

   (iii) Define the conditional probability of an event and state Bayes Theorem. [5 points]

   (iv) Explain what is meant by "marginalization". [5 points]

   Solution: See notes.

2. Suppose we have two bags, labeled A and B. Bag A contains 3 white balls and 1 black ball, bag B contains 1 white ball and 3 black balls. We toss a fair coin and select bag A if it comes up heads and otherwise bag B. From the selected bag we now draw 5 balls, one after another, replacing each ball in the bag after it has been selected (the bag always contains 4 balls each time a ball is drawn). We observe 4 white balls and 1 black ball. What is the probability that we selected bag A ? Hint: use Bayes Rule.

   [20 points]

   Solution: Let E be the event that choose bag A and E^c the event that choose bag B. Let F be the event that we observe 4 white and 1 black balls. We need to calculate P(E|F). By Bayes Rule we know that P(E|F)=P(F|E)P(E)/P(F). We know P(E)=½ and P(F|E)=$\binom{5}{1}(3/4)^4(1/4)$ since the probability of drawing a white ball from bag A is ¾ and a black ball is ¼ and there are five different combinations possible (black ball drawn first, second and so on). So we just need P(F). We have:

   $P(F)=P(F|E)P(E)+P(F|E^c)P(E^c) = \binom{5}{1}(3/4)^4(1/4)(1/2)+\binom{5}{1}(1/4)^4(3/4)(1/2)$

   since $P(F|E^c)=\binom{5}{1}(1/4)^4(3/4)$ and $P(E^c)=1-P(E)= 1/2$. Therefore,

   $P(E|F)= (3/4)^4(1/4)(1/2)/\big((3/4)^4(1/4)(1/2)+(1/4)^4(3/4)(1/2)\big) = 0.96$

3. (i) Define the expected value of a random variable. Give a proof that the expected value is linear i.e. E[X+Y]=E[X]+E[Y] for random variables X and Y.

   [5 points]

   (ii) Define what it means for two random variables to be independent. Give a proof that when two random variables X and Y are independent then E[XY]=E[X]E[Y]. [5 points]

(iii) Define the covariance and correlation of two random variables X and Y.

[5 points]

Solution: See notes.

4. (i) A bag contains 30 balls, of which 10 are red and the other 20 blue. Suppose you take out 8 balls from this bag, with replacement. What is the probability that among the 8 balls in this sample exactly 3 are red and 5 are blue? [5 points]

   Solution. Since we have replacement each draw is independent. Probability of a red is 10/30 and of a blue is 20/30. There are 8!/(5!3!) different permutations in which we can have 3 red and 5 blue balls. So probability of 3 red and 5 blue is $8!/(5!3!)\ (10/30)^3(20/30)^5$.

   (ii) Now suppose that the balls are taken out of the bag without replacement. What is the probability that out of 8 balls exactly 3 are red and 5 are blue?

   [10 points]

   Solution. Since its without replacement the draws are no longer independent. The total number of ways to take 8 balls out of 30 is $\binom{30}{8}$. Picking 3 red balls of 10 can be done $\binom{10}{3}$ ways. Similarly, picking 5 blue balls out of 20 can be done $\binom{20}{5}$ ways. So the probability is $\binom{10}{3}\binom{20}{5}/\binom{30}{8}$.