Reliability Updating in Linear Opinion Pooling for Multiple Decision Makers

Donnacha Bolger & Brett Houlding

Discipline of Statistics, Trinity College Dublin, Ireland.

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We introduce the following notation:

- $P_1, \ldots, P_n$ - a collection of $n$ DMs.
- $\theta$ - some unknown parameter of interest.
- $f_1(\theta), \ldots, f_n(\theta)$ - respective DM beliefs over $\theta$.
- $u_1(r), \ldots, u_n(r)$ - respective DM utility functions.
- $r \in \mathcal{R}$ - a return in the return space.
- $d \in \mathcal{D}$ - a decision in the decision space.
- $d^*$ - a decision deemed optimal.
Neighbours

- Decision quality is based on information quality.
- Inaccurate information $\rightarrow$ bad decisions.
- Accurate information $\rightarrow$ good decisions.
- Important to have information from various diverse sources. Increases a DM's knowledge pool.
- Neighbours.

Idea: Share beliefs with neighbours. Incorporate their beliefs into your own.
Combining Beliefs

How do DMs express their uncertainty? How do they combine beliefs?

- Mathematically vs. Behaviourally.
- Linear Opinion Pooling vs. Bayesian Approach.
- Arithmetic vs. Alternatives.
Linear Arithmetic Opinion Pool

We denote the belief of $P_i$, having received the beliefs of her neighbours, as $\hat{f}_i(\theta)$:

$$\hat{f}_i(\theta) = \alpha_{i,1}f_1(\theta) + \ldots + \alpha_{i,n}f_n(\theta)$$

(1)

Basic properties:

- $\alpha_{i,j}$ - Weight $P_i$ gives to beliefs of $P_j$.
- $\alpha_{i,j} \geq 0$ for all $i, j = 1, \ldots, n$.
- $\sum_{j=1}^{n} \alpha_{i,j} = 1$ for all $i = 1, \ldots, n$.

Big Questions: How do we determine weights? What do we want them to represent?
Optimal Decision

We say the optimal decision for $P_i$ is

$$d^* = \arg \max_{d \in \mathcal{D}} E[u_i(d)] = \int u_i(d, \theta) \hat{f}_i(\theta) d\theta.$$  \hfill (2)

How can we interpret $\hat{f}_i(\theta)$ in this context?

Updating is performed on $f_i(\theta)$ in standard Bayesian manner.

$$f_i(\theta|r) \propto f_i(r|\theta)f_i(\theta)$$  \hfill (3)

It is clear that the weights should be related to reliability. But how? We discuss three approaches here.
Kullback-Leibler

DMs will often express a bias towards beliefs similar to their own. We introduce a measure of this similarity - the KL divergence.

\[ D(f_i||f_j) = \int f_i(\theta) \log \left( \frac{f_i(\theta)}{f_j(\theta)} \right) d\theta \]  

(4)

Basic properties:

- \( D(f_i||f_j) \geq 0 \) for all \( i, j = 1, \ldots, n \).
- \( D(f_i||f_j) = 0 \) if and only if \( f_i(\theta) = f_j(\theta) \) for all \( \theta \in \Theta \).

Question: How can DMs factor these results into their decision process?
KL approach

- **Step 1:** $P_i$ chooses $\alpha_{i,i}$.

- **Step 2:** Each $P_i$ calculates $D(f_i \| f_j)$ for all $j = 1, \ldots, i - 1, i + 1, \ldots, n$.

- **Step 3:**
  \[
  \alpha_{i,j} = \frac{1}{\sum_{j \neq i} D(f_i \| f_j)} \times (1 - \alpha_{i,i}).
  \]
  (5)

- **Step 4:** Form $\hat{f}_i(\theta)$ and make a decision.

- **Step 5:** Observe some return and update $f_i(\theta)$.

- **Step 6:** Repeat.

Advantages and Disadvantages?
Plug-In approach

Initial Equal Weights - Laplacian Principle of Indifference.

- Initial decision made, and outcome $\theta^*$ observed.
- Calculate $f_i(\theta^*)$.
- Let $b_i = \frac{1}{f_i(\theta^*)}$.
- Update weight to $\alpha^*_{i,j}$:

$$\alpha^*_{i,j} = \frac{1}{(\alpha_{i,j})^{-1} + b_i} \sum_{i=1}^{n} \frac{1}{(\alpha_{i,j})^{-1} + b_i}$$

Problems with $f_i(\theta^*) = 0$? Advantages and Disadvantages?
Differing Viewpoints

- This method incorporates both probability and utility in weighting.

- An initial decision is made as in the PI approach.

\[ E_{i|j}[u_i(d^*)] = \int u_i(d^*, \theta) f_j(\theta) d\theta. \]  

\[ w_{i,j} = |u_i(r) - E_{i|j}[u_i(d^*)]|. \]  

\[ \alpha_{i,j}^* = \frac{1}{\sum_{j=1}^{n} \frac{1}{(\alpha_{i,j})^{-1} + w_{i,j}}}. \]  

- Each DM considers the expected utility under the beliefs of each \( P_j \) in turn for the decision that was deemed optimal.

- Nice Properties.
Example Setup

We compare the results of these approaches on a common example.

- $P_1, P_2, P_3, P_4$ must decide whether to enter into a long forward on a stock ($d_1$) or not ($d_2$).
- $\theta$ is a latent parameter pertaining to stock performance.
- Let $\theta \sim \mathcal{N}(\mu, 3)$.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$f_i(\mu)$</th>
<th>$u_i(r)$</th>
<th>Initial fortune</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>$f_1(\mu) \sim \mathcal{N}(5, 1)$</td>
<td>$r$</td>
<td>$50$</td>
</tr>
<tr>
<td>$P_2$</td>
<td>$f_2(\mu) \sim \mathcal{N}(2, 3)$</td>
<td>$r^3$</td>
<td>$45$</td>
</tr>
<tr>
<td>$P_3$</td>
<td>$f_3(\mu) \sim \mathcal{N}(-1, 3)$</td>
<td>$80r - 0.5r^2$</td>
<td>$60$</td>
</tr>
<tr>
<td>$P_4$</td>
<td>$f_4(\mu) \sim \mathcal{N}(-4, 1)$</td>
<td>$e^{\frac{r}{15}}$</td>
<td>$35$</td>
</tr>
</tbody>
</table>
KL approach

- All DMs except $P_4$ opt for $d_1$ at Epoch 1.
- A loss of $\$2$ is observed.
- Beliefs and Weights are updated accordingly.
- All DMs except $P_1$ opt for $d_2$ at Epoch 2.

Here we see the change in beliefs and weights from the perspective of $P_1$. 

![Graph showing change in beliefs and weights from the perspective of P1.](image-url)
PI approach

- All DMs opt for $d_1$ at Epoch 1.
- A loss of $2$ is observed.
- Beliefs and Weights are updated accordingly.
- All DMs opt for $d_2$ at Epoch 2.

Here we see the change in beliefs and weights from the perspective of $P_1$. 
**DV approach**

- All DMs opt for $d_1$ at Epoch 1.
- A loss of $\$2$ is observed.
- Beliefs and Weights are updated accordingly.
- All DMs opt for $d_2$ at Epoch 2.

Here we see the change in beliefs and weights from the perspective of $P_1$. 

![Density distribution graphs](image)
Comparisons

We have approached the same problem using different techniques. What conclusions can we draw?

- KL allows a DM to be “stubborn”. Is this good or bad?
- PI rewards accuracy but is it too extreme?
- DV seems to give the most measured results.
Some directions for further research include:

- A rigorous mathematical justification for use of these approaches.
- Extension to a full group setting.
- A non arbitrary updating rule for the KL approach.
- Incorporation of Non-Parametric techniques.
Thank you very much for listening!

Any questions?