We are working with **natural deduction proofs** $A_1 \ldots A_n \vdash B$ in propositional logic. Deduction rules so far:

→ **Conjunction:**
\[
\frac{A_1 \quad A_2}{A_1 \land A_2} \land i \quad \frac{A_1 \land A_2}{A_1} \land e_1 \quad \frac{A_1 \land A_2}{A_2} \land e_2
\]

→ **Disjunction:**
\[
\frac{A_1}{A_1 \lor A_2} \lor i_1 \quad \frac{A_2}{A_1 \lor A_2} \lor i_2 \quad \frac{A_1 \lor A_2}{B} \lor e
\]

→ **Implication:**
\[
\frac{A}{A \rightarrow B} \rightarrow i \quad \frac{A \rightarrow B}{B} \rightarrow e
\]

and the derived:
\[
\frac{A_1 \rightarrow A_2 \quad \neg A_2}{\neg A_1} \text{ MT}
\]
A NOTE ABOUT IMPLICATION

Notice the introduction rule of implication:

\[
\begin{array}{c}
\text{A} \\
\text{\ldots} \\
\text{B}
\end{array}
\quad \rightarrow i
\]

\[
A \rightarrow B
\]

It contains only one premise:

\[
\begin{array}{c}
\text{A} \\
\text{\ldots} \\
\text{B}
\end{array}
\]
Notice the introduction rule of implication:

\[
\begin{array}{c}
A \\
\ldots \\
B
\end{array} \rightarrow i
\]

\[
A ightarrow B
\]

It contains only one premise:

\[
\begin{array}{c}
A \\
\ldots \\
B
\end{array}
\]

This means that we can think of premises of the form
as premises of the form \(A \rightarrow B\)
A NOTE ABOUT IMPLICATION

Notice the introduction rule of implication:

\[
\frac{A \quad \ldots \quad B}{A \to B} \quad \to i
\]

It contains only one premise:

\[
\begin{array}{c}
A \\
\vdots \\
B
\end{array}
\]

This means that we can think of premises of the form

\[
\begin{array}{c}
A \\
\vdots \\
B
\end{array}
\]

as premises of the form \( A \to B \).

We would have exactly the same logic if we replaced all box-premises with implication-premises, except for the premise in \((\to i)\).
We have seen how to prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$.

Example statement: If it rained then the road is wet. Therefore, if the road is not wet then it did not rain.

Exercise: $p \rightarrow q \rightarrow r, p, p \rightarrow q \vdash r$

\[
\begin{array}{c}
  \boxed{\begin{array}{c}
    A \\
    \ldots \\
    B
  \end{array}} \\
  A \rightarrow B \rightarrow i
  \\
  A \rightarrow A \rightarrow B \\
  B \rightarrow e
  \\
  A_1 \rightarrow A_2 \\
  \neg A_2 \rightarrow MT
  \\
  \neg A_1
\end{array}
\]
Is this entailment correct? \((p \lor q) \rightarrow r \vdash (p \rightarrow r) \land (q \rightarrow r)\)

\[
\begin{align*}
\frac{A_1}{A_1 \land A_2} & \quad \frac{A_1 \land A_2}{A_1} & \quad \frac{A_1 \land A_2}{A_2} \\
& \quad \land i & \quad \land e_1 & \quad \land e_2
\end{align*}
\]

\[
\begin{align*}
\frac{A_1}{A_1 \lor A_2} & \quad \frac{A_2}{A_1 \lor A_2} \\
& \quad \lor i_1 & \quad \lor i_2
\end{align*}
\]

\[
\begin{align*}
\frac{A_1}{A_1 \lor A_2} & \quad \frac{A_2}{A_1 \lor A_2} & \quad \frac{A_1 \lor A_2}{B} \\
& \quad \lor i & \quad \lor i & \quad \lor e
\end{align*}
\]

\[
\begin{align*}
\frac{A}{A_1 \lor A_2} & \quad \frac{A}{A_1 \lor A_2} \\
& \quad \rightarrow i & \quad \rightarrow e & \quad \rightarrow i \quad \rightarrow i
\end{align*}
\]

\[
\begin{align*}
\frac{A \rightarrow B}{A \rightarrow B} \\
& \quad \rightarrow e & \quad \rightarrow e & \quad \rightarrow e
\end{align*}
\]

\[
\begin{align*}
\frac{A_1 \rightarrow A_2}{A_1 \rightarrow A_2} & \quad \frac{\neg A_2}{\neg A_1} \\
& \quad \neg A_1 & \quad \neg A_2
\end{align*}
\]

\[
\begin{align*}
\frac{A_1 \lor A_2}{B} \\
& \quad \lor e
\end{align*}
\]

\[
\begin{align*}
\frac{A_1 \lor A_2}{B} \\
& \quad \lor e
\end{align*}
\]

\[
\begin{align*}
\frac{A_1 \lor A_2}{B} \\
& \quad \lor e
\end{align*}
\]
The following derivable rule has a trivial proof. Sometimes this is useful to prove the goals of inner proofs using the established facts before these proofs.

\[ \frac{A}{A} \text{ COPY} \]

Show the following theorem: \( \vdash p \rightarrow q \rightarrow p \).
Negation
Writing **contradictions** in the logic:

→ $p \land \neg p$
→ $(p \land q) \land \neg(p \land q)$
→ $(p \rightarrow \neg q \lor r) \land \neg(p \rightarrow \neg q \lor r)$
→ ...

---

**Contradictions** are formulas whose semantics returns $F$ for all models. They are unsatisfiable. Formulas of the form $A \land : A$ are unsatisfiable. They are all semantically equivalent: $A \land : A \land : B$, for all $A; B$.

We should be able to prove $A \land : A \vdash \vdash B \land : B$, for all $A; B$.

1. In fact we will show $A \land : A \vdash B$, for all $A; B$!

Intuition: if something as absurd as $A \land : A$ is considered true then any $B$ can be shown to be true.

1. $A \land B$ means $A \land j = B$ and $B \land j = A$;
2. $A \vdash \vdash B$ means $A \vdash B$ and $B \vdash A.$
Writing **contradictions** in the logic:

\[ \rightarrow p \land \neg p \]

\[ \rightarrow (p \land q) \land \neg(p \land q) \]

\[ \rightarrow (p \rightarrow \neg q \lor r) \land \neg(p \rightarrow \neg q \lor r) \]

\[ \rightarrow \ldots \]

Contradictions are formulas whose semantics returns F for all models.
Writing contradictions in the logic:

\[ p \land \neg p \]
\[ (p \land q) \land \neg(p \land q) \]
\[ (p \rightarrow \neg q \lor r) \land \neg(p \rightarrow \neg q \lor r) \]
\[ \ldots \]

Contradictions are formulas whose semantics returns F for all models. They are unsatisfiable.

Formulas of the form \( A \land \neg A \) are unsatisfiable.
Writing contradictions in the logic:

\[ \to p \land \neg p \]
\[ \to (p \land q) \land \neg(p \land q) \]
\[ \to (p \to \neg q \lor r) \land \neg(p \to \neg q \lor r) \]
\[ \to \ldots \]

Contradictions are formulas whose semantics returns \( F \) for all models. They are unsatisfiable.

Formulas of the form \( A \land \neg A \) are unsatisfiable.

They are all semantically equivalent: \( A \land \neg A \equiv B \land \neg B \), for all \( A, B \). \(^1\)

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\(^1\)\( A \equiv B \) means \( A \models B \) and \( B \models A \)
Writing **contradictions** in the logic:

→  $p \land \neg p$
→  $(p \land q) \land \neg(p \land q)$
→  $(p \rightarrow \neg q \lor r) \land \neg(p \rightarrow \neg q \lor r)$
→  ...

Contradictions are formulas whose semantics returns $F$ for all models. They are **unsatisfiable**.

Formulas of the form $A \land \neg A$ are unsatisfiable.

They are all semantically equivalent: $A \land \neg A \equiv B \land \neg B$, for all $A, B$.  

We should be able to prove $A \land \neg A \vdash B \land \neg B$, for all $A, B$.  

---

\(^1\) $A \equiv B$ means $A \models B$ and $B \models A$

\(^2\) $A \not\vdash B$ means $A \nvdash B$ and $B \vdash A$
Writing **contradictions** in the logic:

\[
\begin{align*}
\rightarrow & \quad p \land \neg p \\
\rightarrow & \quad (p \land q) \land \neg (p \land q) \\
\rightarrow & \quad (p \rightarrow \neg q \lor r) \land \neg (p \rightarrow \neg q \lor r) \\
\rightarrow & \quad ...
\end{align*}
\]

Contradictions are formulas whose semantics returns **F** for all models. They are **unsatisfiable**.

Formulas of the form \( A \land \neg A \) are unsatisfiable.

They are all semantically equivalent: \( A \land \neg A \equiv B \land \neg B \), for all \( A, B \).

We should be able to prove \( A \land \neg A \not\vdash B \land \neg B \), for all \( A, B \).

In fact we will show \( A \land \neg A \vdash B \), for all \( A, B \! \)

Intuition: if something as absurd as \( A \land \neg A \) is considered true then any \( B \) can be shown to be true.

---

1. \( A \equiv B \) means \( A \models B \) and \( B \models A \)
2. \( A \not\vdash B \) means \( A \nvdash B \) and \( B \nvdash A \)
We will pick an atomic proposition (say $p$) and name the following:

$\rightarrow$ we write $\bot$ (pronounced “bottom”) to represent $p \land \neg p$

$\rightarrow$ we also write $\top$ (pronounced “top”) to represent $\neg(p \land \neg p)$

(we don’t need the latter here but it will be useful to have later on)
We will pick an atomic proposition (say $p$) and name the following:

1. → we write $\bot$ (pronounced “bottom”) to represent $p \land \neg p$
2. → we also write $\top$ (pronounced “top”) to represent $\neg(p \land \neg p)$ (we don’t need the latter here but it will be useful to have later on)

We will allow to introduce $\bot$ from any contradiction (not just $p \land \neg p$). This rule eliminates $\neg$:

$$\frac{A}{\bot} \neg e$$
We will pick an atomic proposition (say $p$) and name the following:

→ we write $\bot$ (pronounced “bottom”) to represent $p \land \neg p$
→ we also write $\top$ (pronounced “top”) to represent $\neg(p \land \neg p)$

(we don’t need the latter here but it will be useful to have later on)

We will allow to introduce $\bot$ from any contradiction (not just $p \land \neg p$). This rule **eliminates** $\neg$:

$$
\begin{array}{c}
A \\
\neg A \\
\bot
\end{array}
\rightarrow
\neg e
$$

To introduce a negation $\neg A$ we must show that from $A$ we can derive bottom (a contradiction).
We will pick an atomic proposition (say $p$) and name the following:

→ we write $\bot$ (pronounced “bottom”) to represent $p \land \neg p$
→ we also write $\top$ (pronounced “top”) to represent $\neg(p \land \neg p)$

(we don’t need the latter here but it will be useful to have later on)

We will allow to introduce $\bot$ from any contradiction (not just $p \land \neg p$).
This rule eliminates $\neg$:

$$
\begin{array}{c}
A \\
\vdash \neg A \\
\bot \\
\end{array} 
\quad \neg e
$$

To introduce a negation $\neg A$ we must show that from $A$ we can derive bottom (a contradiction).

$$
\begin{array}{c}
\vdash \bot \\
\end{array} 
\quad \neg i
$$
We will pick an atomic proposition (say $p$) and name the following:

→ we write $\bot$ (pronounced “bottom”) to represent $p \land \neg p$
→ we also write $\top$ (pronounced “top”) to represent $\neg(p \land \neg p)$

(we don’t need the latter here but it will be useful to have later on)

We will allow to introduce $\bot$ from any contradiction (not just $p \land \neg p$).
This rule eliminates $\neg$:

\[
\begin{array}{c}
A \\
\neg A \\
\bot
\end{array} \quad \neg e
\]

To introduce a negation $\neg A$ we must show that from $A$ we can derive bottom (a contradiction).

\[
\begin{array}{c}
A \\
\bot
\end{array} \quad \neg e
\]

Finally, from bottom we are allowed to derive anything:

\[
\begin{array}{c}
\bot \\
A
\end{array} \quad \bot e
\]
Show: \[ (p \to \neg q \lor r) \land \neg (p \to \neg q \lor r) \vdash s \]
Show: \[ p, \neg q \vdash \neg(p \rightarrow q) \]