CS2010: ALGORITHMS AND DATA STRUCTURES

Lecture 10: Recursion vs Iteration

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→ **Call stack**: is a stack maintained by the Java runtime system

→ One **call stack frame** (aka activation record) for each **running instance** of a method: contains all information necessary to execute the method
  - **references** to parameter values and local objects, return address etc.

→ **Objects** themselves are stored in another part of memory: the **heap**

→ every time a method is called, a new stack frame is pushed on the call stack.

→ every time a method returns, the top-most stack frame is popped.
Recursion: when something is defined in terms of itself.

Infinite Recursion

Well-founded Recursion
**Principle:** A method is **recursive** when its definition calls the method itself.

A correct recursive method should be **well-founded:** it should terminate/must end up at a **base case**.

**Classic example:** factorial – in math written as: \( n! \)

Math definition:

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0! = 1 \\
(n)! = n \cdot (n-1)! \quad \text{when } n > 0
\]
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- $0! = 1$
- $n! = n \cdot (n - 1)!$ when $n > 0$

**Classic example:** Fibonacci numbers

Math definition:
- $fib(0) = 1$
- $fib(1) = 1$
- $fin(n) = fib(n - 1) + fib(n - 2)$ when $n > 1$
**Principle:** A method is *recursive* when its definition *calls the method itself*. A correct recursive method should be *well-founded*: it should terminate/must end up at a *base case*.

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**Classic example:** Fibonacci numbers

Math definition:
- \( fib(0) = 1 \)
- \( fib(1) = 1 \)
- \( fin(n) = fib(n - 1) + fib(n - 2) \) when \( n > 1 \)

It is convenient to implement recursive math definitions using recursive methods.
Recursive implementation:

```c
int fac(int n) {

}
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```c
int fac(int n) {
    if (n == 0)
        return 1;
    else
        return n * fac(n-1);
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→ Closely matches the mathematical definition.
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  → specify how smaller solutions compose into the solutions of recursive cases
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  → the rest of the cases are the recursive cases (here when \( n > 0 \))
  → specify how smaller solutions compose into the solutions of recursive cases
  → it is usually a top-down calculation
Recursive implementation:

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int fac(int n) {
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    else
        return n * fac(n-1);
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```

→ Q: asymptotic worst-case running time? (# recursive calls * cost of each call)
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→ Q: memory space for call stack frames? (max # frames on call stack)

All this call stack space is needed because of the

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A: $\Theta(n)$ time

→ Q: memory space for call stack frames? (max # frames on call stack)  
A: $\Theta(n)$ space

All this call stack space is needed because of the \texttt{return n * ...}
Recursive implementation using accumulator: (H/W: can you implement the accumulator version bottom-up?)

```c
int fac(int n) { return facAcc(n, 1); }

int facAcc(int n, int acc) {
    if (n == 0)
        return acc;
    else
        return facAcc(n-1, acc * n);
}
```
Recursive implementation using **accumulator**: (H/W: can you implement the accumulator version **bottom-up**?)

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→ Q: asymptotic worst-case running time? \( \Theta(n) \)
→ Q: memory space for call stack frames?
Recursive implementation using **accumulator**: (H/W: can you implement the accumulator version **bottom-up**?)

```java
int fac(int n) { return facAcc(n, 1); }

int facAcc(int n, int acc) {
    if (n == 0)
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        return facAcc(n-1, acc * n);
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```

→ Q: asymptotic worst-case running time? \(\Theta(n)\)

→ Q: memory space for call stack frames?

→ In Java \(\Theta(n)\) for stack space

→ In other, mainly functional, languages (ML, Lisp, Haskell, ...) the compiler runs this using \(\Theta(1)\) stack space.

Only the top-most stack frame is necessary because every function call simply returns the inner result: `return facAcc(n-1, acc * n)`

This is called **tail recursion**
Recursive implementation using accumulator: (H/W: can you implement the accumulator version bottom-up?)

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    → In Java \( \Theta(n) \) for stack space
    → In other, mainly functional, languages (ML, Lisp, Haskell, ...) the compiler runs this using \( \Theta(1) \) stack space.
Only the top-most stack frame is necessary because every function call simply returns the inner result: `return facAcc(n-1, acc * n)`
This is called tail recursion
From tail recursive implementation → iterative implementation:

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int fac(int n) { return facAcc(n, 1); }

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}
```

```c
int fac(int n) {
    int acc = 1;
    for ( ; !(n == 0); n--) {
        acc = acc * n;
    }
    return acc;
}
```
From tail recursive implementation → iterative implementation:

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}
```

→ Running time of iterative implementation: $\Theta(n)$
From tail recursive implementation $\rightarrow$ iterative implementation:

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$\rightarrow$ Running time of iterative implementation: $\mathcal{O}(n)$

$\rightarrow$ Stack space of iterative implementation: $\mathcal{O}(1)$

In functional languages this simple translation is done by the compiler!
FIBONACCI NUMBERS

Math definition:

\[
\begin{align*}
\text{fib}(0) &= 1 \\
\text{fib}(1) &= 1 \\
\text{fin}(n) &= \text{fib}(n - 1) + \text{fib}(n - 2) \quad \text{when } n > 1
\end{align*}
\]

Recursive implementation:
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Recursive implementation:

```c
int fib(int n) {
    if (n <= 1) return 1;
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→ Running time: non-tight upper bound: \(O(2^n)\) tight bound: \(\Theta(fib(n))\)

→ # of recursive calls of fib(n) is the size of the binary tree of recursive calls with \(n\) levels (\(\leq 2^n\)); however this is not a complete tree; thus \(O(2^n)\).
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→ Call stack space: \(\Theta(n)\)
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Recursive implementation with accumulator (tail recursion):

```java
int fib(int n) { return fibAcc(n, 1, 1); }

int fibAcc(int n, int last, int secondToLast) {
    if (n <= 1) return last;
    else return fibAcc(n-1, last + secondToLast, last);
}
```

→ This is much trickier! Read it again off-line and understand why it works.

→ It is a bottom-up calculation using two accumulators.
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Math definition:

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→ This is much trickier! Read it again off-line and understand why it works.
→ It is a bottom-up calculation using two accumulators.
→ Running time: \( \Theta(n) \)
→ Call stack space: \( \Theta(n) \) in Java and \( \Theta(1) \) in other languages.
Tail-recursive implementation —→ iterative implementation

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}
```

→ Worst Case Asymptotic Running time: $\Theta(n)$
→ Call stack space: $\Theta(1)$

```c
int fib(int n) {
  int last = 1; int secondToLast = 1;
  for ( ; !(n <= 1); n--) {
    int tmpLast = last;
    last = last + secondToLast;
    secondToLast = tmpLast;
  }
  return last;
}
```
**GREATEST COMMON DIVISOR (GCD)**

$\text{gcd}(x, y)$ is the largest number $n$ such that $x \% n = y \% n = 0$.

For simplicity assume $x \leq y$. 
GREATEST COMMON DIVISOR (GCD)

\( \text{gcd}(x,y) \) is the largest number \( n \) such that \( x \% n = y \% n = 0 \).

For simplicity assume \( x \leq y \).

**Attempt 1**: try all numbers \( n \) from \(+\text{inf}\) down to 1 until you find one that has the above property.
\textbf{GREATEST COMMON DIVISOR (GCD)}

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For simplicity assume \( x \leq y \).

\textbf{Attempt 1:} try all numbers \( n \) from \(+\text{inf}\) down to 1 until you find one that has the above property.

\textbf{Attempt 2:} try all numbers \( n \) from \( x \) down to 1 until you find one that has the above property (because \( \text{gcd} \) will necessarily be \( \leq \text{min}(x, y) \)).
GREATEST COMMON DIVISOR (GCD)

\( \text{gcd}(x,y) \) is the largest number \( n \) such that \( x \% n = y \% n = 0 \).

For simplicity assume \( x \leq y \).

**Attempt 3: Euclid’s algorithm** a Divide & Conquer approach:

**Euclid’s Theorem:** \( \text{gcd}(x,y) = \text{gcd}(y, x \% y) \).

The base case here is \( \text{gcd}(x,0) = x \) (why is this the base case?).

→ Q: How many iterations in the worst case?
greatest common divisor (gcd) is the largest number \( n \) such that \( x\%n = y\%n = 0 \).

For simplicity assume \( x \leq y \).

**Attempt 3: Euclid’s algorithm** a **Divide & Conquer** approach:

**Euclid’s Theorem:** \( \text{gcd}(x,y) = \text{gcd}(y, x \% y) \).

The base case here is \( \text{gcd}(x,0) = x \) (why is this the base case?).

→ **Q:** How many iterations in the worst case?

→ **Gabriel Lame’s Theorem (1844):** \#iterations < 5 \cdot h

Where \( h = \text{digits of min}(x,y) \) (here this is \( x \)) in base 10.

→ **A:** \( O(lg(x)) \)
It depends on the benefits vs the drawbacks each implementation gives us in each case.

Recursive algorithms usually have easier proofs of correctness (Divide and Conquer is more obvious and similar to mathematical induction).

Iterative algorithms usually have easier proofs of memory usage (and in Java use less memory).

Recursive algorithms are sometimes easier to implement when we have complex data structures, because the compiler maintains a stack of previous calls for us.

We will use recursive implementations for most operations over trees. We could have used recursive implementations for operations over lists & arrays but:

they are not simpler than the iterative implementations

Java will need $O(n)$ call stack space to execute them.
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  → they are not simpler than the iterative implementations
  → Java will need $O(n)$ call stack space to execute them.
Homework 1: Implement Euclid’s algorithm using

→ A recursive method.
→ An iterative method.

Homework 2: give recursive implementations for:

→ binary search over an array
→ linear search over a linked list

Homework 3: Implement search on a Binary Search Tree (next lecture) using recursion and compare with the iterative version in the book.

Homework 4**: give an iterative implementation for method put on binary search tree.