Concurrent Strategies

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The next-generation domain theory? An intensional theory to capture the ways of computing, to near operational concerns and reasoning? An event-based theory?

A new result characterizing (nondeterministic) concurrent strategies
Representations of traditional domains

What is the information order? What are the ‘units’ of information?
Two answers:

(‘Topological’) [Scott]: Propositions about finite properties;
more information corresponds to more propositions being true.
Functions are ordered pointwise. Can represent domains via logical theories
(‘Information systems’, ‘Logic of domains’).

(‘Temporal’) [Berry]: Events (atomic actions);
more information corresponds to more events having occurred.
Intensional ‘stable order’ on ‘stable’ functions. (‘Stable domain theory’)
Can represent Berry’s dl domains as event structures.
**Event structures**

An *event structure* comprises \((E, \text{Con}, \leq)\), consisting of a set of *events* \(E\) - partially ordered by \(\leq\), the *causal dependency relation*, and - a nonempty family \(\text{Con}\) of finite subsets of \(E\), the *consistency relation*, which satisfy

\[
\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,
\]

\[
\{e\} \in \text{Con} \text{ for all } e \in E,
\]

\[
Y \subseteq X \in \text{Con} \Rightarrow Y \in \text{Con}, \quad \text{and}
\]

\[
X \in \text{Con} \& e \leq e' \in X \Rightarrow X \cup \{e\} \in \text{Con}.
\]

Say \(e, e'\) are *concurrent* if \(\{e, e'\} \in \text{Con} \& e \not\sim e' \& e' \not\sim e\).

In games the relation of *immediate* dependency \(e \rightarrow e'\), meaning \(e\) and \(e'\) are distinct with \(e \leq e'\) and no event in between, will play a very important role.
Configurations of an event structure

The configurations, $C^\infty(E)$, of an event structure $E$ consist of those subsets $x \subseteq E$ which are

**Consistent:** $\forall X \subseteq_{\text{fin}} x. \ X \in \text{Con}$ and

**Down-closed:** $\forall e, e'. e' \leq e \in x \Rightarrow e' \in x$.

For an event $e$ the set $[e] =_{\text{def}} \{e' \in E \mid e' \leq e\}$ is a configuration describing the whole causal history of the event $e$.

$x \subseteq x'$, i.e. $x$ is a sub-configuration of $x'$, means that $x$ is a sub-history of $x'$.

If $E$ is countable, $(C^\infty(E), \subseteq)$ is a dI domain (and all such are so obtained). Here concentrate on the finite configurations $C(E)$. 

Event structures as types, e.g., Streams as event structures

conflict (inconsistency)  \rightarrow  \text{immed. causal dependency}
Simple parallel composition

000 \ ~~ \ 001 \ 010 \ ~~ \ 011 \ 110 \ ~~ \ 111

00 \ ~~ \ 01 \ : \ ~~~ \ 11

0 \ ~~~ \ 1

aaa \ ~~ \ aab \ aba \ ~~ \ abb \ bba \ ~~~ \ bbb

aa \ ~~ \ ab \ : \ ~~~ \ bb

a \ ~~~ \ b
Event structures as processes

- Semantics of synchronising processes [Hoare, Milner] can be expressed in terms of universal constructions on event structures, and other models.
- Relations between models via adjunctions.

In this context, a simulation map of event structures \( f : E \to E' \) is a partial function on events \( f : E \to E' \) such that for all \( x \in C(E) \)

\[
fx \in C(E') \quad \text{and} \\
\text{if } e_1, e_2 \in x \text{ and } f(e_1) = f(e_2), \text{ then } e_1 = e_2. \quad ('event linearity')
\]

**Idea:** the occurrence of an event \( e \) in \( E \) induces the coincident occurrence of the event \( f(e) \) in \( E' \) whenever it is defined.
Process constructions on event structures

“Partial synchronous” product: $A \times B$ with projections $\Pi_1$ and $\Pi_2$, cf. CCS synchronized composition where all events of $A$ can synchronize with all events of $B$. (Hard to construct directly so use e.g. stable families.)

Restriction: $E \upharpoonright R$, the restriction of an event structure $E$ to a subset of events $R$, has events $E' = \{ e \in E \mid [e] \subseteq R \}$ with causal dependency and consistency restricted from $E$.

Synchronized compositions: restrictions of products $A \times B \upharpoonright R$, where $R$ specifies the allowed synchronized and unsynchronized events.

Projection: Let $E$ be an event structure. Let $V$ be a subset of ‘visible’ events. The projection of $E$ on $V$, $E\downarrow V$, has events $V$ with causal dependency and consistency restricted from $E$.

[Event structures as types and processes? Spans]
Product—an example
Concurrent games

Basics

Games and strategies are represented by *event structures with polarity*. The two polarities $+$ and $-$ express the dichotomy:

- player/opponent;
- process/environment;
- ally/enemy.

An *event structure with polarity* is one in which all events carry a polarity $+/−$, respected by maps.

**Dual**, $E^\perp$, of an event structure with polarity $E$ is a copy of the event structure $E$ with a reversal of polarities; $\overline{e} \in E^\perp$ is complement of $e \in E$, and *vice versa*.

A (nondeterministic) concurrent *pre-strategy* in game $A$ is a total map $\sigma : S \to A$ of event structures with polarity.
Pre-strategies

A pre-strategy $\sigma : A \leftrightarrow B$ is a total map of event structures with polarity

$$\sigma : S \rightarrow A^\perp \parallel B.$$  

It determines a span of event structures with polarity

$$
\begin{array}{c}
\sigma_1 \quad S \\ \\
A^\perp & \sigma_2 \\ \\
& B
\end{array}
$$

where $\sigma_1, \sigma_2$ are partial maps of event structures with polarity; one and only one of $\sigma_1, \sigma_2$ is defined on each event of $S$. 
Composing pre-strategies

Two pre-strategies $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$ as spans:

\[
\begin{array}{ccc}
A^\perp & \xrightarrow{\sigma_1} & S & \xrightarrow{\sigma_2} & B \\
& & & & \downarrow \tau_1 \\
& & & & B^\perp \xrightarrow{\tau_2} C
\end{array}
\]

Their composition

\[
\begin{array}{ccc}
A^\perp & \xrightarrow{(\tau \circ \sigma)_1} & T \circ S & \xrightarrow{(\tau \circ \sigma)_2} & C
\end{array}
\]

where $T \circ S =_{\text{def}} (S \times T \upharpoonright \text{Syn}) \downarrow \text{Vis}$ where ...
Their composition: $T \circ S = \text{def} \ (S \times T \upharpoonright \text{Syn}) \downarrow \text{Vis}$ where

\[
\text{Syn} = \{ p \in S \times T \mid \sigma_1 \Pi_1(p) \text{ is defined} \} \cup \{ p \in S \times T \mid \sigma_2 \Pi_1(p) = \tau_1 \Pi_2(p) \text{ with both defined} \} \cup \{ p \in S \times T \mid \tau_2 \Pi_2(p) \text{ is defined} \},
\]

\[
\text{Vis} = \{ p \in S \times T \upharpoonright \text{Syn} \mid \sigma_1 \Pi_1(p) \text{ is defined} \} \cup \{ p \in S \times T \upharpoonright \text{Syn} \mid \tau_2 \Pi_2(p) \text{ is defined} \}.
\]
Concurrent copy-cat

Identities on games $A$ are given by copy-cat strategies $\gamma_A : \mathcal{CC}_A \to A_\perp \parallel A$ —strategies for player based on copying the latest moves made by opponent.

$\mathcal{CC}_A$ has the same events, consistency and polarity as $A_\perp \parallel A$ but with causal dependency $\leq_{\mathcal{CC}_A}$ given as the transitive closure of the relation

$$\leq_{A_\perp \parallel A} \cup \{(\overline{c}, c) \mid c \in A_\perp \parallel A \& \text{pol}_{A_\perp \parallel A}(c) = +\}$$

where $\overline{c} \leftrightarrow c$ is the natural correspondence between $A_\perp$ and $A$. The map $\gamma$ is the identity on the common underlying set of events. Then,

$$x \in \mathcal{C}(\mathcal{CC}_A) \text{ iff } x \in \mathcal{C}(A_\perp \parallel A) \& \forall c \in x. \text{pol}_{A_\perp \parallel A}(c) = + \Rightarrow \overline{c} \in x.$$
Copy-cat—an example

\[ \text{CC}_A \]

\[ A \]

\[ \overline{a}_2 \]

\[ \overline{a}_1 \]

\[ a_2 \]

\[ a_1 \]
Theorem characterizing concurrent strategies

Receptivity $\sigma : S \rightarrow A^\perp \parallel B$ is receptive when $\sigma(x) \subseteq^- y$ implies there is a unique $x' \in C(S)$ such that $x \subseteq x'$ & $\sigma(x') = y$.

\[ x \xrightarrow{\subseteq^-} x' \]
\[ \sigma(x) \subseteq^- y \]

Innocence $\sigma : S \rightarrow A^\perp \parallel B$ is innocent when it is

+-Innocence: If $s \rightarrow s'$ & $\text{pol}(s) = +$ then $\sigma(s) \rightarrow \sigma(s')$ and

--Innocence: If $s \rightarrow s'$ & $\text{pol}(s') = -$ then $\sigma(s) \rightarrow \sigma(s')$.

[\rightarrow\text{ stands for immediate causal dependency}]

Theorem Receptivity and innocence are necessary and sufficient for copy-cat to act as identity w.r.t. composition: $\sigma \circ \gamma_A \cong \sigma$ and $\gamma_B \circ \sigma \cong \sigma$ for all $\sigma : A \rightarrow\leftarrow B$. 
The bicategory of concurrent games

Definition A strategy is a receptive, innocent pre-strategy.

A bicategory, \textbf{Games}, whose

- **objects** are event structures with polarity—the games,
- **arrows** are strategies \( \sigma : A \leftrightarrow B \)
- **2-cells** are maps of spans.

The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies \( \odot \) (which extends to a functor on 2-cells via the functoriality of synchronized composition).
Deterministic strategies

Say an event structures with polarity $S$ is *deterministic* iff

$$
\forall X \subseteq_{\text{fin}} S. \text{Neg}[X] \in \text{Con}_S \Rightarrow X \in \text{Con}_S,
$$

where $\text{Neg}[X] = \text{def} \ \{ s' \in S \mid \exists s \in X. \text{pol}_S(s') = - \& s' \leq s \}$.

Say a strategy $\sigma : S \to A$ is deterministic if $S$ is deterministic.

**Proposition** An event structure with polarity $S$ is deterministic iff $s \rightarrow s'$ implies $x \cup \{ s, s' \} \in \mathcal{C}(S)$, for all $x \in \mathcal{C}(S)$.

**Notation** $x \rightarrow e \subseteq y$ iff $x \cup \{ e \} = y \& e \notin x$, for configurations $x, y$, event $e$.

$x \rightarrow e \subseteq$ iff $\exists y. x \rightarrow e \subseteq y$. 
Nondeterministic copy-cats

(i) Take $A$ to consist of two $+$ve events and one $-$ve event, with any two but not all three events consistent. The construction of $\mathbb{C}_A$:

\[
\begin{align*}
\emptyset & \rightarrow \emptyset \\
A^\perp & \emptyset \rightarrow \emptyset \ A \\
\emptyset & \leftarrow \emptyset
\end{align*}
\]

(ii) Take $A$ to consist of two events, one $+$ve and one $-$ve event, inconsistent with each other. The construction $\mathbb{C}_A$:

\[
\begin{align*}
A^\perp & \emptyset \rightarrow \emptyset \ A \\
\emptyset & \leftarrow \emptyset
\end{align*}
\]
Lemma Let $A$ be an event structure with polarity. The copy-cat strategy $\gamma_A$ is deterministic iff $A$ satisfies

$$\forall x \in C(A). \ x \xrightarrow{a} \subset \& \ x \xrightarrow{a'} \subset \& \ \text{pol}_A(a) = + \ & \text{pol}_A(a') = - \ \Rightarrow \ x \cup \{a, a'\} \in C(A). \quad (\exists)$$

Lemma The composition $\tau \circ \sigma$ of two deterministic strategies $\sigma$ and $\tau$ is deterministic.

Proposition A deterministic strategy $\sigma : S \rightarrow A$ is injective on configurations (equivalently, $\sigma : S \twoheadrightarrow A$).

$\leadsto$ sub-bicategory $\mathbf{DGames}$, equivalent to an order-enriched category.
Theorem A subfamily $F \subseteq \mathcal{C}(A)$ has the form $\sigma \mathcal{C}(S)$ for a deterministic strategy $\sigma : S \rightarrow A$, iff

reachability: $\emptyset \in F$ and if $x \in F$, $\emptyset \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} x_k = x$ within $F$;

determinacy: If $x \xrightarrow{a} \subset$ and $x \xrightarrow{a'} \subset$ in $F$ with $\text{pol}_A(a) = +$, then $x \cup \{a, a'\} \in F$;

receptivity: If $x \in F$ and $x \xrightarrow{a} \subset$ in $\mathcal{C}(A)$ and $\text{pol}_A(a) = -$, then $x \cup \{a\} \in F$;

$+\text{-innocence}$: If $x \xrightarrow{a} \subset x_1 \xrightarrow{a'} \subset$ & $\text{pol}_A(a) = +$ in $F$ & $x \xrightarrow{a'} \subset$ in $\mathcal{C}(A)$, then $x \xrightarrow{a'} \subset$ in $F$ (receptivity implies $\text{innocence}$);

cube: In $F$, $x \xrightarrow{a} \xleftarrow{e} y_1 \xrightarrow{b} \cdots$ implies

\[
\begin{array}{ccc}
x & \xrightarrow{a} & x_1 \xleftarrow{e} y_1 \xrightarrow{b} \cdots \\
& b & \xrightarrow{a} & y \xrightarrow{e} z \\
& b & \xrightarrow{a} & x_2 \xleftarrow{e} y_2 \xrightarrow{a} & \\
\end{array}
\quad
\begin{array}{ccc}
x & \xrightarrow{a} & x_1 \xleftarrow{e} y_1 \xrightarrow{b} \cdots \\
& e & \xrightarrow{a} & w \xrightarrow{b} z \\
& b & \xrightarrow{a} & x_2 \xleftarrow{e} y_2 \xrightarrow{a} & \\
\end{array}
\]
Related work—early results

**Stable spans, profunctors and stable functions** The sub-bicategory of $\text{Games}$ where the events of games are purely $+ve$ is equivalent to the bicategory of stable spans:

![Diagram](image)

where $S^+$ is the projection of $S$ to its $+ve$ events; $\sigma_2^+$ is the restriction of $\sigma_2$ to $S^+$ is rigid; $\sigma_2^-$ is a demand map taking $x \in C(S^+)$ to $\sigma_2^-(x) = \sigma_1^+[x]$. Composition of stable spans coincides with composition of their associated profunctors.

When deterministic (and event structures are countable) we obtain a sub-bicategory equivalent to Berry’s **dl-domains and stable functions**.
Related work continued

**Ingenuous strategies** Deterministic concurrent strategies coincide with the *receptive ingenuous* strategies of and Melliès and Mimram.

**Closure operators** A deterministic strategy $\sigma: S \rightarrow A$ determines a closure operator $\varphi$ on $C^\infty(S)$: for $x \in C^\infty(S)$,

$$\varphi(x) = x \cup \{ s \in S \mid pol(s) = + \& Neg\{s\} \subseteq x \}.$$ 

The closure operator $\varphi$ on $C^\infty(S)$ induces a *partial* closure operator $\varphi_p$ on $C^\infty(A)$ and in turn a closure operator $\varphi_p^\top$ on $C^\infty(A)^\top$.

**Simple games** “Simple games” [Hyland *et al.*] arise when we restrict Games to objects and deterministic strategies in $\mathcal{PA}^-\#$ — alternating games, with conflicting branches, beginning with opponent moves.
Categories for games

Adjunctions

\[
\begin{array}{c}
\mathcal{PA}_r \overset{\top}{\longrightarrow} \mathcal{PF}_r \overset{\top}{\longrightarrow} \mathcal{PE}_r \overset{\top}{\longrightarrow} \mathcal{PE}_t \\
\mathcal{PA}_t^\# \overset{\top}{\longrightarrow} \mathcal{PA}_t^\# \overset{\top}{\longrightarrow} \mathcal{PF}_t^\#
\end{array}
\]

Conway games inhabit \( \mathcal{PF}_t^\# = \mathcal{PF}_r^\# \), a coreflective subcategory of \( \mathcal{PE}_t \). Conway’s ‘sum’ is obtained by applying the right adjoint to their \( \|\)-composition in \( \mathcal{PE}_t \).

‘Simple games’ belong to \( \mathcal{PA}_r^\# \), “polarized” games, starting with moves of Opponent. ‘Tensor’ of simple games got by applying the right adjoint of \( \mathcal{PA}_t^\# \hookrightarrow \mathcal{PE}_t \) to their \( \|\)-composition in \( \mathcal{PE}_t \).
Current problems:

Recovering games with copying. *E.g.*, can the (co)monads for Hyland-Ong games be got from (co)monads on $P\mathcal{E}_t$ with symmetry?

In special cases, strategies can be transformed s.t. composition of strategies can be expressed as the usual composition of spans. I don’t think this is so in general?
ERC Project:
The next-generation semantics involves causal models, also becoming important in a range of areas from security, systems, model checking, systems biology, to proof theory.

ECSYM: Events, Causality and Symmetry—the next-generation semantics

Objective 1 Intensional semantics: games; strong correspondence with operational semantics; metalanguage(s); higher-dimensional algebra; names

Objective 2 Event-based reasoning: event types; event induction; causal reasoning; program logics (“Reynolds’ conjecture” for conc. sepn. logic); names

Objective 3 Quantitative reasoning: probabilistic; stochastic; quantum(?)

Objective 4 Application methods: security; rule-based systems biology; distributed algorithms; extending SOS to causal models