Incremental semantic scales by strings

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scales: linear order . . .

semantic:

$$\texttt{the soup } \underbrace{\texttt{cool}}_{\textit{temperature}}\texttt{ed in/for an } \underbrace{\texttt{hour}}_{\textit{time}}$$

incremental: bounded span and granularity

strings: finite, discrete — based on a finite alphabet

Why category theory?

months in a year $\fbox{Jan}\fbox{Feb}\cdots\fbox{Dec}$

+ days $\fbox{Jan,d1}\fbox{Jan,d2}\cdots\fbox{Jan,d31}\fbox{Feb,d1}\cdots\fbox{Dec,d31}$

The real line $\mathbb{R}$
- time (Kamp, W. Klein, van Lambalgen & Hamm, . . .)
- degrees (Universal Density of Measurement, Fox & Hackl)

Increment strings via morphisms (arrows) in a category

Dense linear orders via inverse limit
- functor between categories

Dependent types + morphisms f

$(\prod i \in I)\ F(i)$ $\qquad f(a(j)) = a(i) \in F(i)$ for $i \to j$ in I

$(\sum i \in I)\ F(i)$ $\qquad$ Grothendieck construction $\int F$

Temporal structure simplified

W. Klein

> *The expression of time in natural languages relates a clause-internal temporal structure to a clause-external temporal structure.*
> *The latter may shrink to a single interval, for example, the time at which the sentence is uttered; but this is just a special case. The clause-internal temporal structure may also be very simple – it may be reduced to a single interval without any further differentiation, the 'time of the situation'; but if this ever happens, it is only a borderline case.*
> *As a rule, the clause-internal structure is much more complex.*

`Ed explained` $\boxed{\text{E}}\boxed{\text{S}}$

Hypothesis: the semantics of tense and aspect is finite-state

More strings

`soup cool in an hour` $\boxed{x, d \leq sDg}\boxed{d \leq sDg}\boxed{\text{hour}(x), sDg < d}$

`soup cool for an hour` $\boxed{x}\boxed{[\sqsupseteq]sDg_{\downarrow}}\boxed{\text{hour}(x), [\sqsupseteq]sDg_{\downarrow}}$

`Ed explaining` $\boxed{\text{E}} \overset{\text{V}}{\rightsquigarrow} \boxed{\text{E}_{\circ}}\boxed{\text{E}_{\circ},\text{V}}\boxed{\text{E}_{\circ}}$

	Reichenbach	tense	aspect
`it rained`	$\boxed{\text{E,R}}\boxed{\text{S}}$	$\boxed{\text{R}}\boxed{\text{S}}$	$\boxed{\text{E,R}}$
`it has rained`	$\boxed{\text{E}}\boxed{\text{R,S}}$	$\boxed{\text{R,S}}$	$\boxed{\text{E}}\boxed{\text{R}}$

Projection $\boxed{\text{E,R}}\boxed{\text{S}} \rightsquigarrow \boxed{\text{R}}\boxed{\text{S}}$

Superposition $\boxed{\text{E,R}}\boxed{\text{S}} = \boxed{\text{R}}\boxed{\text{S}} \;\&\; \boxed{\text{E,R}}\boxed{}$

Constraints - telic, homogeneous . . .

Three projections

days in a year $\rightsquigarrow$ months in a year

$$\boxed{\text{Jan,d1}}\boxed{\text{Jan,d2}}\cdots\boxed{\text{Dec,d31}} \overset{\rho_{months}}{\rightsquigarrow} \boxed{\text{Jan}}^{31}\boxed{\text{Feb}}^{28}\cdots\boxed{\text{Dec}}^{31}$$

$$\overset{bc}{\rightsquigarrow} \boxed{\text{Jan}}\boxed{\text{Feb}}\cdots\boxed{\text{Dec}}$$

ρ_A "see only A"

$$\rho_A(\alpha_1 \cdots \alpha_n) \ := \ (\alpha_1 \cap A) \cdots (\alpha_n \cap A)$$

bc "no time without change"
compress α^+ to α [no stutters / identical adjacent boxes]

unpad no initial or final empty boxes

$$unpad(bc(\rho_{\{\text{Feb}\}}(s))) = \boxed{\text{Feb}}$$

$$unpad(bc(\rho_{\{\text{d3}\}}(s))) = (\boxed{\text{d3}}\boxed{})^{11}\boxed{\text{d3}}$$

Projecting satisfaction

$$\text{hole}(a) \ := \ \overset{\boxed{a}\boxed{}\boxed{a}}{(\exists x, y, z)(x < y < z \wedge U_a(x) \wedge U_a(z) \wedge \neg U_a(y))}$$

$$\text{interval}(a) \ := \ \exists x(U_a(x) \wedge \neg \text{hole}(a))$$

$$\boxed{a'}\boxed{a, a', a''}\boxed{a''} \models \exists x(U_a(x) \wedge U_{a'}(x))$$

$$\text{iff} \quad \underbrace{\rho_{\{a,a'\}}(\boxed{a'}\boxed{a, a', a''}\boxed{a''})}_{\boxed{a'}\boxed{a, a'}\boxed{}} \models \underbrace{\exists x(U_a(x) \wedge U_{a'}(x))}_{\in sen(\{a, a'\})}$$

For $s \in (2^B)^+$ and $\varphi \in sen(A)$ with $A \subseteq B$,

$$s \models \varphi \quad \text{iff} \quad \rho_A(s) \models \varphi$$

$$s \models \text{interval}(a) \quad \text{iff} \quad unpad(bc(\rho_{\{a\}}(s))) = \boxed{a}$$

The presheaf $\mathbf{P}_f$ on $Fin(\Phi)$

Approximate an infinite set Φ via its set $Fin(\Phi)$ of finite subsets.

For $A \subseteq B \in Fin(\Phi)$,

$$\begin{aligned}\mathbf{P}_f(A) &:= \{f(s) \mid s \in (2^A)^+\} \\ \mathbf{P}_f(B,A) &: \mathbf{P}_f(B) \to \mathbf{P}_f(A) \\ & \qquad s \mapsto f(\rho_A(s)) =: f_A(s)\end{aligned}$$

For f equal to id or bc or $(bc; unpad)$,

$$\mathbf{P}_f(A,A) = id_{\mathbf{P}_f(A)} \quad \text{– i.e.,} \quad (\forall s \in (2^A)^+)\ f_A(f(s)) = f(s)$$

and for $A \subseteq B \subseteq C \in Fin(\Phi)$,

$$\mathbf{P}_f(C,A) = \mathbf{P}_f(C,B); \mathbf{P}_f(B,A) \quad \text{– i.e.,} \quad (\forall s \in (2^C)^+)\ f_A(f(s)) = f_A(f_B(f(s)))$$

Satisfaction via institutions (Goguen & Burstall 1992)

whenever $A \subseteq B \in Fin(\Phi)$, $\varphi \in sen(A)$ and $s \in \mathbf{P}_f(B)$,

$$s \models_B \varphi \quad \text{iff} \quad f_A(s) \models_A \varphi$$

for $f = id$, $\rho_A(s)$ encodes the A-reduct of the model s encodes

- $Fin(\Phi), \subseteq$ is the *signature* category **Sign**
- $\mathbf{P}_f : Fin(\Phi)^{op} \to \mathbf{Set}$ is the *model* presheaf *Mod* on **Sign**
- *sentence* functor $sen : Fin(\Phi) \to \mathbf{Set}$
  $sen(A,B) : sen(A) \hookrightarrow sen(B)$ for $A \subseteq B$

Monadic Second-Order Logic (MSO) and $\mathbf{P}_f(A, \varphi)$

Strings in MSO/Büchi-Elgot-Trakhtenbrot are over A, not 2^A

$$aa' \;\mapsto\; \boxed{a \mid a'} \;\in\; (2^{\{a,a'\}})^+$$

$\exists x(U_a(x) \wedge U_{a'}(x))$ is satisfiable not in A^+ but in $(2^A)^+$

many automata, only partially known, on different clocks

Let $\mathbf{P}_f(A, \varphi) \;:=\; \{s \in \mathbf{P}_f(A) \mid s \models_A \varphi\}$

$L \subseteq (2^A)^+$ is regular iff $L = \mathbf{P}_{id}(A, \varphi)$ for some φ in MSO_A

$\mathbf{P}_f(A, \varphi)$ is regular for f computable by a finite-state transducer

Step from $\mathbf{P}_{id}$ to $\mathbf{P}_f$ for dense linear orders in $\varprojlim \mathbf{P}_{bc}$
whereas $\varprojlim \mathbf{P}_{id}$ keeps lengths fixed (discrete)

Approximations and open-endedness

Ranta 1994 on the need to "justify everyday objects in type theory"

> *mathematical and logical reasoning is based on* fully presented objects. *A natural number is fully presented by its canonical expression. The structure of this expression determines it as a natural number. But there is no fully presenting expression for the continent of Africa, say. Even the longest encyclopedic text will leave an infinity of properties open, and it is a puzzling question what expression, if any, would determine Africa as a continent.*
>
> *... develop the techniques of* approximating *full presentations of objects*

Open-ended Φ approximated via $Fin(\Phi)$
$\leadsto$ $\mathbf{P}_f$ $\leadsto$ finite-state methods

Inverse limit as IL

$$\varprojlim \mathbf{P}_f \quad := \quad \{w \in \prod_{A \in Fin(\Phi)} \mathbf{P}_f(A) \mid w(A) = f_A(w(B))$$
$$\text{whenever } A \subseteq B \in Fin(\Phi)\}$$

Given $A \in Fin(\Phi)$ and $\varphi \in sen(A)$, let

$$[\![A, \varphi]\!]_f \quad := \quad \{w \in \varprojlim \mathbf{P}_f \mid w(A) \models_A \varphi\}$$

and analyze entailment

$$(A, \varphi) \vdash (A', \varphi') \quad \text{iff} \quad [\![A, \varphi]\!]_f \subseteq [\![A, \varphi]\!]_f$$

in terms of $\int \mathbf{Q}_f$, where

$$\mathbf{Q}_f(A) \;:=\; \{\mathbf{P}_f(A, \varphi) \mid \varphi \in sen(A)\}$$
$$\mathbf{Q}_f(B, A) \;:\; \mathbf{Q}_f(B) \to \mathbf{Q}_f(A), \quad \mathbf{P}_f(B, \varphi) \mapsto \mathbf{P}_f(A, \varphi)$$

Continuous change

`soup cool in an hour` $\boxed{x, d \leq sDg \mid d \leq sDg \mid \text{hour}(x), sDg < d}$

$$=_{bc} \quad \boxed{d \leq sDg}^{+} \boxed{sDg < d} \;\&\; \boxed{x \mid \;}^{+} \boxed{\text{hour}(x)}$$

$$I \models d \leq sDg \;\text{ iff }\; (\forall r \in I)\; d \leq sDg(r)$$

`soup cool for an hour` $\boxed{x \mid [\sqsupseteq] sDg_{\downarrow} \mid \text{hour}(x), [\sqsupseteq] sDg_{\downarrow}}$

$$=_{bc} \quad \boxed{\; \mid [\sqsupseteq] sDg_{\downarrow}}^{+} \;\&\; \boxed{x \mid \;}^{+} \boxed{\text{hour}(x)}$$

$$sDg_{\downarrow} \quad := \quad \exists x(sDg < x \wedge \text{Prev}(x \leq sDg))$$

$$I \models [\sqsupseteq]\varphi \;\text{ iff }\; (\forall I' \sqsubseteq I)\; I' \models \varphi$$

Projecting satisfaction, again

For $a \in A \in Fin(\Phi)$ and $s \in (2^A)^+$,

$$s \models_A \text{interval}(a) \quad \text{iff} \quad unpad(bc(\rho_{\{a\}}(s))) = \boxed{a}$$

where

$$\begin{aligned} \text{interval}(a) &:= \exists x(U_a(x) \wedge \neg \text{hole}(a)) \\ \text{hole}(a) &:= (\exists x, y, z)(x < y < z \wedge U_a(x) \wedge U_a(z) \wedge \neg U_a(y)) \end{aligned}$$

For $A \subseteq B \in Fin(\Phi)$ and $s \in (2^B)^+$,

$$s \models_B \text{Spec}(A) \quad \text{iff} \quad bc(\rho_A(s)) \in \{\boxed{a} \mid a \in A\}^+$$

where

$$\text{Spec}(A) := \forall x \bigvee_{a \in A} (U_a(x) \wedge \bigwedge_{a' \in A - \{a\}} \neg U_{a'}(x))$$

Frames as automata

(1) Jones did it slowly, deliberately, in the bathroom, with a knife, at midnight. (Davidson 1967)

(2) $\begin{bmatrix} agent = \text{JONES} \\ how = \begin{bmatrix} \textbf{slow} \\ \textbf{deliberate} \end{bmatrix} \\ where = \begin{bmatrix} \textbf{bathroom} \end{bmatrix} \\ when = \begin{bmatrix} \textbf{midnight} \end{bmatrix} \\ with\text{-}what = \begin{bmatrix} \textbf{knife} \end{bmatrix} \end{bmatrix}$

(3) $\begin{bmatrix} l_1 = r_1 \\ \vdots \\ l_k = r_k \end{bmatrix} \qquad r \overset{l_i}{\rightsquigarrow} r_i$

- set L of labels l
- relations $\overset{l}{\rightsquigarrow} \subseteq R \times R$ for $l \in L$

Records as regular languages

The *trace set* $tr_{\leadsto}(r)$ *of* r is

$$\bigcup_{k\geq 1}\{l_1\cdots l_k \in L^k \mid (\exists r_1\cdots r_k \in R^k)\ r \overset{l_1}{\leadsto} r_1 \text{ and } r_{i-1} \overset{l_i}{\leadsto} r_i \text{ for } 1 < i \leq k\}$$

Fact 1 (Myhil-Nerode theorem). $tr_{\leadsto}(r)$ is a regular language iff

$$\{tr_{\leadsto}(r') \mid r \leadsto_{+} r'\} \text{ is finite}$$

(where $\leadsto_{+}$ is the transitive closure of $\bigcup_l \overset{l}{\leadsto}$).

Fact 2. For deterministic transitions, trace equivalence

$$r \sim_{tr} r' \quad \text{iff} \quad tr_{\leadsto}(r) = tr_{\leadsto}(r')$$

is a bisimulation (a $\leadsto$-congruence).

But what about record types?

Proposal. Use signatures in institutions

$$\mathbf{P}_f(A) \quad \leadsto \quad \mathbf{P}_f\,\underbrace{(A,\varphi)}_{\text{expanded signature}}$$

Expand

$$\begin{aligned} sen(A,\varphi) \quad = \quad & sen(A) \cup \\ & \{\psi \mid (A,\varphi) \text{ satisfies } \psi\text{'s presuppositions}\} \end{aligned}$$

$\ldots$ future work?