Prior probabilities of Allen interval relations over finite orders

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$\mathcal{AR}$: a widely used basis for relating (temporal) intervals
(e.g., Liu et al 2018, Verhagen et al 2009)

Hans Kamp: discourse time (from events)

> *when we interpret a piece of discourse — or a single sentence in the context in which it is being used — we build something like a model of the episode or situation described; and an important part of that model are its event structure, and the time structure that can be derived from that event structure*

ISO-TimeML (Pustejovsky, Lee, Bunt, ...): TLINK tags

Plan

§1 Allen interval relations

§2 Probabilities over n ordered points

§3 Probabilities over n interval names

§4 Conclusion

Allen relations as strings (Schwer, Durand)

$$a \;\approx (l(a), r(a)] \approx \{x \mid l(a) < x \leq r(a)\}$$

$$ama' \;:\; l(a) < r(a) = l(a') < r(a') \qquad \boxed{l(a)}\boxed{r(a), l(a')}\boxed{r(a')}$$

$$\mathfrak{s}_{\mathrm{m}}(a, a') \;:=\; \boxed{a}\boxed{a, a'}\boxed{a'}$$

$$aba' \;:\; l(a) < r(a) < l(a') < r(a') \qquad \boxed{l(a)}\boxed{r(a)}\boxed{l(a')}\boxed{r(a')}$$

R	$\mathfrak{s}_R(a, a')$	R	$\mathfrak{s}_R(a, a')$	R	$\mathfrak{s}_R(a, a')$
b	$\boxed{a}\boxed{a}\boxed{a'}\boxed{a'}$	m	$\boxed{a}\boxed{a, a'}\boxed{a'}$	o	$\boxed{a}\boxed{a'}\boxed{a}\boxed{a'}$
d	$\boxed{a'}\boxed{a}\boxed{a}\boxed{a'}$	s	$\boxed{a, a'}\boxed{a}\boxed{a'}$	f	$\boxed{a'}\boxed{a}\boxed{a, a'}$
e	$\boxed{a, a'}\boxed{a, a'}$				

$$\mathfrak{s}_{R^{-1}}(a, a') = \mathfrak{s}_R(a', a)$$

Allen's transitivity table

$$t(R_1, R_2) \;:=\; \{R \in \mathcal{AR} \mid \text{for some order with intervals } a, a', a'',\; aR_1a',\; a'R_2a'' \text{ and } aRa''\}$$

e.g. $t(\mathrm{b}, \mathrm{b}) = \{\mathrm{b}\}$ $\qquad t(\mathrm{o}, \mathrm{d}) = \{\mathrm{d}, \mathrm{o}, \mathrm{s}\}$ $\qquad t(\mathrm{b}, \mathrm{bi}) = \mathcal{AR}$

$$\#(R) \;:=\; \sum_{R' \in \mathcal{AR}} \mathrm{card}(t(R, R')) \;=\; \sum_{R' \in \mathcal{AR}} \mathrm{card}(t(R', R))$$

$$= \begin{cases} 41 & \text{if length}(\mathfrak{s}_R) = 4 \quad \text{(long: b,d,o,bi,di,oi)} \\ 25 & \text{if length}(\mathfrak{s}_R) = 3 \quad \text{(medium: m,s,f,mi,si,fi)} \\ 13 & \text{if length}(\mathfrak{s}_R) = 2 \quad \text{(short: e)} \end{cases}$$

Plan

Probabilities defined

$$
\begin{aligned}
[n] &:= \{1, 2, \ldots, n\} \\
\Omega_n &:= \{f : \{x, y, x', y'\} \to [n] \mid f(x) < f(y) \text{ and } f(x') < f(y')\}
\end{aligned}
$$

$$
f \text{ satisfies } R \iff (f(x), f(y)]\ R\ (f(x'), f(y')]
$$

$$
p_n(R) = \frac{\operatorname{card}(\{f \in \Omega_n \mid f \text{ satisfies } R\})}{\operatorname{card}(\Omega_n)}.
$$

where for $n \geq 4$, $\operatorname{card}(\{f \in \Omega_n \mid f \text{ satisfies } R\})$ is

$$
\begin{aligned}
\binom{n}{2} &= \frac{n(n-1)}{2} && \text{if } R \text{ is e} \\
\binom{n}{3} &= \binom{n}{2}\frac{n-2}{3} && \text{if } R \text{ is medium} \\
\binom{n}{4} &= \binom{n}{3}\frac{n-3}{4} && \text{if } R \text{ is long}
\end{aligned}
$$

Probabilities calculated

For $n \geq 4$ and $R, R' \in \mathcal{AR}$,

$$p_n(R) = p_n(R') \text{ if } \text{length}(\mathfrak{s}_R) = \text{length}(\mathfrak{s}_{R'})$$

and

$$\text{card}(\Omega_n) = \binom{n}{2} \cdot \binom{n}{2}$$

whence

$$\begin{aligned} p_n(\mathrm{e}) &= \frac{2}{n(n-1)} \\ p_n(R) &= \frac{2(n-2)}{3n(n-1)} && \text{for medium } R \\ p_n(R) &= \frac{(n-3)(n-2)}{6n(n-1)} && \text{for long } R \end{aligned}$$

Some probabilities

$$\lim_{n\to\infty} p_n(R) = \begin{cases} 0 & \text{if } R \text{ is short or medium} \\ 1/6 & \text{otherwise} \end{cases}$$

n	$p_n(\mathrm{e})$	$p_n(\mathrm{m})$	$p_n(\mathrm{b})$
4	$1/6$	$1/9$	$1/36$
5	$1/10$	$1/10$	$1/20$
6	$1/15$	$4/45$	$1/15$
8	$1/28$	$1/14$	$5/56$

Plan

Consistent interval labelings as strings

$$\mathcal{L}_n := \{s \in (2^{[n]} - \{\Box\})^+ \mid \text{each } i \in [n] \text{ occurs exactly twice in } s\}$$
$$\mathcal{L}_2 = \{\mathfrak{s}_R(1,2) \mid R \in \mathcal{AR}\}$$

$$\pi_A(\alpha_1 \cdots \alpha_n) := (\alpha_1 \cap A) \cdots (\alpha_n \cap A) \text{ and then delete any } \Box$$
$$\pi_{\{2,3\}}(\boxed{1,2,4}\boxed{1}\boxed{2,3}\boxed{3}\boxed{4}) = \boxed{2}\boxed{2,3}\boxed{3}$$

$$i \text{ occurs exactly twice in } s \iff \pi_{\{i\}}(s) = \boxed{i}\boxed{i}$$
$$s \models iRi' \iff \pi_{\{i,i'\}} = \mathfrak{s}_R(i,i')$$

$f : [n] \times [n] \to \mathcal{AR}$ is *consistent* if for some $s \in \mathcal{L}_n$,

$$(\forall i \in [n])(\forall i' \in [n])\ \pi_{\{i,i'\}}(s) = \mathfrak{s}_{f(i,i')}(i,i')$$

Probabilities defined

Fact.

(i) For all $s \in \mathcal{L}_n$ and $(i, i') \in [n] \times [n]$, there is a unique $R \in \mathcal{AR}$ s.t. $\pi_{\{i,i'\}}(s) = \mathfrak{s}_R(i, i')$.

(ii) The map $s \mapsto \omega_s$ is a bijection from $\mathcal{L}_n$ onto the set of consistent labellings from $[n] \times [n]$ to $\mathcal{AR}$, where $\omega_s : [n] \times [n] \to \mathcal{AR}$ sends (i, i') to the unique $R \in \mathcal{AR}$ s.t. $\pi_{\{i,i'\}}(s) = \mathfrak{s}_R(i, i')$.

$$\mathcal{L}_n(R) \;:=\; \{s \in \mathcal{L}_n \mid \pi_{\{1,2\}}(s) = \mathfrak{s}_R(1, 2)\}$$

$$p_n(R) \;:=\; \frac{\mathrm{card}(\mathcal{L}_n(R))}{\mathrm{card}(\mathcal{L}_n)}$$

Calculate $\mathrm{card}(\mathcal{L}_n(R))$ and $\mathrm{card}(\mathcal{L}_n)$ through superposition

Superposition

$$\&(\boxed{i}\boxed{i},\ \boxed{i'}\boxed{i'},\ s) \iff s \in \{\mathfrak{s}_R(i, i') \mid R \in \mathcal{AR}\}.$$

$$\text{(i0)}\ \frac{}{\&(\epsilon, \epsilon, \epsilon)} \qquad \text{(i1)}\ \frac{\&(s, s', s'')}{\&(s\alpha, s'\alpha', s''(\alpha \cup \alpha'))}$$

$$\text{(i2)}\ \frac{\&(s, s', s'')}{\&(s\alpha, s', s''\alpha)} \qquad \text{(i3)}\ \frac{\&(s, s', s'')}{\&(s, s'\alpha', s''\alpha')}$$

$$\overset{\text{(i0)}}{\leadsto}\ (\epsilon, \epsilon, \epsilon) \overset{\text{(i2)}}{\leadsto} (\boxed{i},\ \epsilon, \boxed{i}) \overset{\text{(i1)}}{\leadsto} (\boxed{i}\boxed{i},\ \boxed{i'},\ \boxed{i}\boxed{i, i'})$$

$$\overset{\text{(i3)}}{\leadsto} (\boxed{i}\boxed{i},\ \boxed{i'}\boxed{i'},\ \boxed{i}\boxed{i, i'}\boxed{i'})$$

$$L \& L' \;:=\; \{s'' \mid (\exists s \in L)(\exists s' \in L')\ \&(s, s', s'')\}$$

A commutative monoid

$$\begin{aligned}\mathcal{L}_1 &= \boxed{1}\boxed{1} \\ \mathcal{L}_{n+1} &= \mathcal{L}_n \ \& \ \boxed{n+1}\boxed{n+1} \qquad \text{for } n \geq 1\end{aligned}$$

$$\begin{aligned}\mathcal{L}_2(R) &= \mathfrak{s}_R(1,2) \\ \mathcal{L}_{n+1}(R) &= \mathcal{L}_n(R) \ \& \ \boxed{n+1}\boxed{n+1} \qquad \text{for } n \geq 2\end{aligned}$$

Given a string s of length $k > 1$, the set $s\& \boxed{n}\boxed{n}$ consists of

- $\binom{k}{2}$ strings of length k,
- $k(k+1)$ strings of length $k+1$, and
- $\binom{k+1}{2} + k + 1$ strings of length $k+2$

Cardinalities of $\mathcal{L}_n(R)$ and $\mathcal{L}_n$

$$\begin{aligned}c_n(R;k) &:= \operatorname{card}(\{s \in \mathcal{L}_n(R) \mid \operatorname{length}(s) = k\}) \\ c_2(R;k) &= \begin{cases} 1 & \text{if } \operatorname{length}(\mathfrak{s}_R) = k \\ 0 & \text{otherwise} \end{cases} \\ c_{n+1}(R;k) &= \frac{k(k-1)}{2}(c_n(R;k) + 2c_n(R;k-1) + c_n(R;k-2))\end{aligned}$$

$$\begin{aligned}\operatorname{card}(\mathcal{L}_n(\mathrm{e})) &= \sum_{k=2}^{2n-2} c_n(\mathrm{e};k) \\ \operatorname{card}(\mathcal{L}_n(R)) &= \sum_{k=3}^{2n-1} c_n(R;k) \quad \text{for medium } R \\ \operatorname{card}(\mathcal{L}_n(R)) &= \sum_{k=4}^{2n} c_n(R;k) \quad \text{for long } R \\ \operatorname{card}(\mathcal{L}_n) &= \operatorname{card}(\mathcal{L}_n(\mathrm{e})) + 6(\operatorname{card}(\mathcal{L}_n(\mathrm{m})) + \operatorname{card}(\mathcal{L}_n(\mathrm{b})))\end{aligned}$$

$\mathrm{card}(\mathcal{L}_n(R))/\mathrm{card}(\mathcal{L}_n)$ for some n

n	$p_n(\mathrm{e})$	$p_n(\mathrm{m})$	$p_n(\mathrm{b})$	$1 - 6p_n(\mathrm{b})$
2	$1/13$	$1/13$	$1/13$	$7/13 \approx 0.538461538$
3	0.031784841	0.061124694	0.100244499	0.398533007
10	0.002527761	0.021841026	0.144404347	0.133573915
100	0.000023782	0.002283051	0.164379652	0.013722086
500	0.000000959	0.000460405	0.166206102	0.002763387
1000	0.000000240	0.000230840	0.166435786	0.001385281
1500	0.000000107	0.000153893	0.166512755	0.000923468

$$p_2(R) = \frac{1}{13} \qquad \text{uniform distribution}$$

$$p_3(R) = \frac{\#(R)}{\sum_{R' \in \mathcal{AR}} \#(R')} \qquad \text{transitivity table}$$

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Models and probabilities

HANS KAMP: discourse time (from events)

> *when we interpret a piece of discourse — or a single sentence in the context in which it is being used — we build something like a model of the episode or situation described; and an important part of that model are its event structure, and the time structure that can be derived from that event structure by means of Russell's construction.*

(MLN) $$p(x) \;=\; \frac{1}{Z}\, exp\Big(\sum_{\varphi \in I} w_\varphi n_\varphi(x)\Big)$$

- finite set I of f-o formulas φ and weights $w_\varphi \in \mathbb{R}$
- $n_\varphi(x)$ is the number of x-groundings satisfying φ

uniform if $\{\varphi \in I \mid w_\varphi \neq 0\} = \emptyset$ (data-free)

(Here) probability of aRa', for *arbitrary* intervals a, a' $(R \in \mathcal{AR})$

Stretches of time vs moments of change

Russell-instant = maximal subset of overlapping events

$$\boxed{a, a'} + \boxed{a}\boxed{a'} + \boxed{a'}\boxed{a}$$

$+$ *pre*, *post* for all Allen relations on a, a' — e.g.,

$$\boxed{a}\boxed{a, a'}\boxed{a'} \;\rightsquigarrow\; \boxed{a, pre(a')}\boxed{a, a'}\boxed{post(a), a'}$$

$$\boxed{a}\boxed{}\boxed{a'} \;\rightsquigarrow\; \boxed{a, pre(a')}\boxed{post(a), pre(a')}\boxed{post(a), a'}$$

$$\underbrace{(}_{l}\ \text{interior}\ \underbrace{]}_{r}$$

$$\underbrace{\text{open-ended interiors}}_{\text{states}} \quad \text{vs} \quad \underbrace{\text{bounding borders}}_{\text{events (dynamic)}}$$

- analyze in Monadic Second-Order Logic (MSO) over strings

Leibniz' law (identity of indiscernibles) & projections

$$x \neq y \;\supset\; (\exists P)\neg(P(x) \equiv P(y)) \qquad \text{(LL)}$$

- take P from a finite set A

$$\begin{aligned} x \not\equiv_A y \;&:=\; \bigvee_{a \in A} \neg(P_a(x) \equiv P_a(y)) & bc \\ &\equiv\; \bigvee_{a \in A} (\neg P_a(x) \wedge P_a(y)) \vee (P_a(x) \wedge \neg P_a(y)) \\ &\qquad\qquad P_{l(a)}(x) \qquad\qquad P_{r(a)}(x) & d_\square \end{aligned}$$

- replace $\neq$ by adjacency S "time steps$_S$ only with change$_A$"

$$x S y \;\supset\; x \not\equiv_A y \qquad (\text{LL}_{S,A})$$

T h a n k   Y o u

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