WHAT JUSTIFIES A LOGICAL LAW?

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Formal Logic
We started the course with a connection between logic and reasoning; that logic was the study of good reasoning, in particular what makes reasoning good.
Logic and reasoning

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Tubby is a teddybear that hugs every child
Therefore, all children are hugged by someone

The liquid in this bottle turns litmus paper red
Therefore, the liquid in this bottle is acidic.

Both are examples of good reasoning, but we were only concerned with examples such as the one to the left: that argument is a case of good reasoning for formal reasons.
A logical law, such as **Modus Ponens** is a law for good reasoning: but what does that mean?

**Modus Ponens (MP)**

If $p \supset q$ and $p$, then $q$. 

A logical law, such as Modus Ponens is a law for good reasoning: but what does that mean?

Modus Ponens (MP)
If $p \supset q$ and $p$, then $q$.

(A) A logical law is one that provides normativity for our reasoning. There is something compelling about the tought that one ought to reason validly, and in accordance with logic more generally.
A logical law, such as Modus Ponens is a law for good reasoning: but what does that mean?

Modus Ponens (MP)
If \( p \supset q \) and \( p \), then \( q \).

(A) A logical law is one that provides normativity for our reasoning. There is something compelling about the thought that one ought to reason validly, and in accordance with logic more generally.

(B) A logical law is one that provides certainty for our reasoning. Whereas other methods provide plausibility and probability, logic establishes its results without the possibility of doubt.
Logic and reasoning

- A logical law, such as **Modus Ponens** is a law for good reasoning: but what does that mean?

  Modus Ponens (MP)
  
  If $p \supset q$ and $p$, then $q$.

(A) A logical law is one that provides **normativity** for our reasoning. There is something compelling about the thought that one ought to reason validly, and in accordance with logic more generally.

(B) A logical law is one that provides **certainty** for our reasoning. Whereas other methods provide plausibility and probability, logic establishes its results without the possibility of doubt.

As always, though, things are not that simple.
Once upon a time...
Once upon a time...

Achilles had overtaken the Tortoise, and had seated himself comfortably on its back. “So you’ve got to the end of our race-course?” said the Tortoise. “Even though it *does* consist of an infinite series of distance? I thought some wiseacre or other had proved that the thing couldn’t be done?”
Once upon a time...

Achilles had overtaken the Tortoise, and had seated himself comfortably on its back. "So you’ve got to the end of our race-course?" said the Tortoise. "Even though it does consist of an infinite series of distance? I thought some wiseacre or other had proved that the thing couldn’t be done?"

"It can be done," said Achilles. "It has be done! Solvitur ambulando. You see, the distances were constantly diminishing; and so —" "But if they had been constantly increasing?" The Tortoise interrupted. "How then?". "Then I shouldn’t be here," Achilles modestly replied; "and you would have got several times round the world, by this time!"
Once upon a time... 

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“It can be done,” said Achilles. “It has be done! Solvitur ambulando. You see, the distances were constantly diminishing; and so —” “But if they had been constantly increasing?” The Tortoise interrupted. “How then?”.

“Then I shouldn’t be here,” Achilles modestly replied; “and you would have got several times round the world, by this time!”

“You flatter me – flatten, I mean,” said the Tortoise; “for you are a heavy weight, and no mistake! Well now, would you like to hear of a race-course most people fancy they can get to the end of in two or three steps, while really consists of an infinite number of distances, each no longer than the previous one?”
So, Achilles, I understand you’ve been learning logic.
The story of Achilles and the Tortoise

T  So, Achilles, I understand you’ve been learning logic.
A  Oh yes, I can now think myself around anybody.
T  So, Achilles, I understand you’ve been learning logic.
A  Oh yes, I can now think myself around anybody.
T  Hmm...yes, I have no doubt that you can think yourself into circles.
The story of Achilles and the Tortoise

T  So, Achilles, I understand you’ve been learning logic.
A  Oh yes, I can now think myself around anybody.
T  Hmm...yes, I have no doubt that you can think yourself into circles.
A  What do you mean by that, you sneaky tortoise?
The story of Achilles and the Tortoise

T So, Achilles, I understand you’ve been learning logic.
A Oh yes, I can now think myself around anybody.
T Hmm...yes, I have no doubt that you can think yourself into circles.
A What do you mean by that, you sneaky tortoise?
T Nothing, nothing. Please, show me something you’ve learned.
The story of Achilles and the Tortoise

If Achilles fights in the battle, the Greeks will win
Achilles fights in the battle
---------------------
The Greeks will win

T So, Achilles, I understand you’ve been learning logic.
A Oh yes, I can now think myself around anybody.
T Hmm...yes, I have no doubt that you can think yourself into circles.
A What do you mean by that, you sneaky tortoise?
T Nothing, nothing. Please, show me something you’ve learned.
A Gladly! [He puffed up his chest] Let me dazzle you with this argument.
The story of Achilles and the Tortoise

If Achilles fights in the battle, the Greeks will win
Achilles fights in the battle

The Greeks will win

T So, Achilles, I understand you’ve been learning logic.
A Oh yes, I can now think myself around anybody.
T Hmm...yes, I have no doubt that you can think yourself into circles.
A What do you mean by that, you sneaky tortoise?
T Nothing, nothing. Please, show me something you’ve learned.
A Gladly! [He puffed up his chest] Let me dazzle you with this argument.
T Now, now, let’s do this logically.
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

(C) \( q \)

T So, Achilles, I understand you’ve been learning logic.
A Oh yes, I can now think myself around anybody.
T Hmm. . .yes, I have no doubt that you can think yourself into circles.
A What do you mean by that, you sneaky tortoise?
T Nothing, nothing. Please, show me something you’ve learned.
A Gladly! [He puffed up his chest] Let me dazzle you with this argument.
T Now, now, let’s do this logically.
A Certainly! Here you go.
The story of Achilles and the Tortoise

(P2) \( p \supset q \)

(P1) \( p \)

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(C) \( q \)

T: That’s impressive, I’m sure, but what’s your point?

Well, do you accept the premises (P2) and (P1)?

Yes, given what they mean I guess I have to.

Well, then you must accept (C) too.

Quod erat demonstrandum.

Latin, Achilles? I thought you were Greek. Why do I have to accept (C)?

Because you accept (P2) and (P1).

And?

Well, if (P2) and (P1) then (C), of course!
The story of Achilles and the Tortoise

\[(P2) \ p \supset q\]
\[(P1) \ p\]
\[
\overset{\text{T}}{\text{That's impressive, I'm sure, but what's your point?}}
\]
\[
\overset{\text{A}}{\text{Well, do you accept the premises (P2) and (P1)?}}
\]
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

\[ \quad \]

(C) \( q \)

T That’s impressive, I’m sure, but what’s your point?
A Well, do you accept the premises (P2) and (P1)?
T Yes, given what they mean I guess I have to.
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

\[ \text{Therefore, } q \]

(T) That’s impressive, I’m sure, but what’s your point?
(A) Well, do you accept the premises (P2) and (P1)?
(T) Yes, given what they mean I guess I have to.
(A) Well, then you must accept (C) too. *Quod erat demonstrandum.*
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

\[ \begin{array}{c}
\text{That's impressive, I'm sure, but what's your point?}
\end{array} \]

A Well, do you accept the premises (P2) and (P1)?

T Yes, given what they mean I guess I have to.

A Well, then you must accept (C) too. *Quod erat demonstrandum.*

T Latin, Achilles? I thought you were Greek. Why do I have to accept (C)?
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

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(C) \( q \)

T That’s impressive, I’m sure, but what’s your point?
A Well, do you accept the premises (P2) and (P1)?
T Yes, given what they mean I guess I have to.
A Well, then you must accept (C) too. Quod erat demonstrandum.
T Latin, Achilles? I thought you were Greek. Why do I have to accept (C)?
A Because you accept (P2) and (P1).
(P2) $p \supset q$
(P1) $p$

\[\text{(C)} q\]

T That’s impressive, I’m sure, but what’s your point?
A Well, do you accept the premises (P2) and (P1)?
T Yes, given what they mean I guess I have to.
A Well, then you \textit{must} accept (C) too. \textit{Quod erat demonstrandum}.
T Latin, Achilles? I thought you were Greek. Why do I have to accept (C)?
A Because you accept (P2) and (P1).
T And?
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

(C) \( q \)

T: That’s impressive, I’m sure, but what’s your point?
A: Well, do you accept the premises (P2) and (P1)?
T: Yes, given what they mean I guess I have to.
A: Well, then you must accept (C) too. \textit{Quod erat demonstrandum.}
T: Latin, Achilles? I thought you were Greek. Why do I have to accept (C)?
A: Because you accept (P2) and (P1).
T: And?
A: Well, if (P2) and (P1) then (C), of course!
The story of Achilles and the Tortoise

(P2) $p \supset q$
(P1) $p$

__________

(C) $q$

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

----------

(C) \( q \)

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
The story of Achilles and the Tortoise

(P2) \( p \supset q \)
(P1) \( p \)

---------------------

(C) \( q \)

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
T Well, that’s a very important premise isn’t it?
The story of Achilles and the Tortoise

(P2) $p \supset q$
(P1) $p$

-------------------

(C) $q$

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
T Well, that’s a very important premise isn’t it?
A Hmph… fine, but then you better accept it. [He scribbled it down]
The story of Achilles and the Tortoise

(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

________________________
(C) \(q\)

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
T Well, that’s a very important premise isn’t it?
A Hmph... fine, but then you better accept it. [He scribbled it down]
The story of Achilles and the Tortoise

(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[\text{(C)}\ q\]

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
T Well, that’s a very important premise isn’t it?
A Hmph. . . fine, but then you better accept it. [He scribbled it down]
T I Promise. Everything logic says is worth writing down, you know.
The story of Achilles and the Tortoise

(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[
\begin{align*}
\text{C) } q
\end{align*}
\]

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
T Well, that’s a very important premise isn’t it?
A Hmph... fine, but then you better accept it. [He scribbled it down]
T I Promise. Everything logic says is worth writing down, you know.
A Now, will you finally shut up and accept (C)?
The story of Achilles and the Tortoise

(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\(\hline\)

(C) \(q\)

T Yes, that sounds fair. I’ll accept it as soon as you note it down.
A Say what?
T Well, that’s a very important premise isn’t it?
A Hmph... fine, but then you better accept it. [He scribbled it down]
T I Promise. Everything logic says is worth writing down, you know.
A Now, will you finally shut up and accept (C)?
T What? Why should I accept it again?

\((p \& (p \supset q)) \supset q\)
Yes, that sounds fair. I’ll accept it as soon as you note it down.

A Say what?

T Well, that’s a very important premise isn’t it?

A Hmph… fine, but then you better accept it. [He scribbled it down]

T I Promise. Everything logic says is worth writing down, you know.

A Now, will you finally shut up and accept (C)?

T What? Why should I accept it again?

A It’s not just your shell that’s thick: if (P3), (P2) and (P1) then (C)!
The story of Achilles and the Tortoise

(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[\text{\textbf{(C)}} \quad q\]

\[\text{T If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.}\]
The story of Achilles and the Tortoise

(P3) $(p \land (p \supset q)) \supset q$
(P2) $p \supset q$
(P1) $p$

__________________________

(C) $q$

T If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.
A What do you mean by that?
The story of Achilles and the Tortoise

(P3) \((p \land (p \supset q)) \supset q\)

(P2) \(p \supset q\)

(P1) \(p\)

___________________________

(C) \(q\)

T If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.

A What do you mean by that?

T I’ll accept that once you’ve noted it down as a premise.
The story of Achilles and the Tortoise

(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[\text{(C)}\ q\]

T If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.
A What do you mean by that?
T I’ll accept that once you’ve noted it down as a premise.
A Have it your way [with a touch of sadness he did]
The story of Achilles and the Tortoise

(P4) \(((p \land (p \supset q)) \land ((p \land (p \supset q)) \supset q)) \supset q\)
(P3) \((p \land (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[\begin{align*}
\text{(C)} & \quad q \\
\text{T} & \quad \text{If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.} \\
\text{A} & \quad \text{What do you mean by that?} \\
\text{T} & \quad \text{I’ll accept that once you’ve noted it down as a premise.} \\
\text{A} & \quad \text{Have it your way [with a touch of sadness he did]} \\
\end{align*}\]
The story of Achilles and the Tortoise

(P4) \(((p \land (p \supset q)) \land ((p \land (p \supset q)) \supset q)) \supset q\)
(P3) \((p \land (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

________________________

(C) \(q\)

T If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.
A What do you mean by that?
T I’ll accept that once you’ve noted it down as a premise.
A Have it your way [with a touch of sadness he did]
T So, what was I had to accept again?
The story of Achilles and the Tortoise

(P4) \(((p \land (p \supset q)) \land ((p \land (p \supset q)) \supset q)) \supset q\)
(P3) \((p \land (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[\begin{align*}
\text{(C)} & \quad q \\
\text{T} & \quad \text{If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.} \\
\text{A} & \quad \text{What do you mean by that?} \\
\text{T} & \quad \text{I'll accept that once you've noted it down as a premise.} \\
\text{A} & \quad \text{Have it your way [with a touch of sadness he did]} \\
\text{T} & \quad \text{So, what was I had to accept again?} \\
\text{A} & \quad \text{Well, because. . . sigh, I guess I understand why they call you } Taught-us\end{align*}\]
(P4) \(((p \& (p \supset q)) \& ((p \& (p \supset q)) \supset q)) \supset q\)
(P3) \((p \& (p \supset q)) \supset q\)
(P2) \(p \supset q\)
(P1) \(p\)

\[\begin{align*}
\text{(C) } q
\end{align*}\]

T If (P3), (P2) and (P1) then (C). . .hmm. Yes, I think I could accept that.
A What do you mean by that?
T I’ll accept that once you’ve noted it down as a premise.
A Have it your way [with a touch of sadness he did]
T So, what was I had to accept again?
A Well, because. . . sigh, I guess I understand why they call you \textit{Taught-us}
T And you, \textit{A Kill-Ease}
The story of Achilles and the Tortoise

\[ \uparrow \text{And so on} \]

(P4) \(((p \land (p \supset q)) \land ((p \land (p \supset q)) \supset q)) \supset q \]

(P3) \(p \land (p \supset q) \supset q \]

(P2) \(p \supset q \]

(P1) \(p \]

\[
\begin{align*}
\text{(C)} & \quad q \\
\text{T} & \quad \text{If (P3), (P2) and (P1) then (C). \ldots hmm. Yes, I think I could accept that.} \\
\text{A} & \quad \text{What do you mean by that?} \\
\text{T} & \quad \text{I'll accept that once you’ve noted it down as a premise.} \\
\text{A} & \quad \text{Have it your way [with a touch of sadness he did]} \\
\text{T} & \quad \text{So, what was I had to accept again?} \\
\text{A} & \quad \text{Well, because.\ldots sigh, I guess I understand why they call you Taugh-us} \\
\text{T} & \quad \text{And you, A Kill-Ease} 
\end{align*}
\]
The plot thickens

Susan Haack
A puzzle about justification
Susan Haack (*The Justification of Deduction*):

Hume presented us with a dilemma: we cannot justify induction deductively, because to do so would be to show that *whenever* the premisses of an inductive argument are true, the conclusion must be true too – which would be *too strong*; and we cannot justify induction inductively, either, because such a ’justification’ would be *circular*. 
The plot thickens: induction

How does an inductive argument work?

**Premises** \((P_{a_1} \& Q_{a_2}), \ldots (P_{a_n} \& Q_{a_n})\)

**Conclusion** \((\forall x)(P_x \supset Q_x)\)
How does an inductive argument work?

**Premises**: 

\((P_1 \land Q_2), \ldots (P_n \land Q_n)\)

**Conclusion**: 

\((\forall x)(P_x \supset Q_x)\)

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The plot thickens: induction
The plot thickens: induction

How does an inductive argument work?

**Premises**: \((Pa_1 \& Qa_2), \ldots (Pa_n \& Qa_n)\)

**Conclusion**: \((\forall x)(Px \supset Qx)\)

**Induction**

**Inductive Justification**

**Deductive Justification**

**IF** Inductive justification then the justification is circular: we assume that we can employ an inductive argument.
How does an inductive argument work?

**Premises**

\[(P_{a_1} \land Q_{a_2}), \ldots (P_{a_n} \land Q_{a_n})\]

**Conclusion**

\[(\forall x)(P_x \supset Q_x)\]

---

**Induction**

**Inductive Justification**

IF **Inductive justification** then the justification is circular: we assume that we can employ an inductive argument.

**Deductive Justification**

IF **Deductive justification** then the justification is too strong: the truth of the premises doesn’t necessitate truth of conclusion.
Susan Haack (*The Justification of Deduction*):

I propose another dilemma: we cannot justify deduction inductively, because to do so would be, at best, to show that *usually*, when the premisses of a deductive argument are true, the conclusion is true too – which would be *too weak*; and we cannot justify deduction deductively, either, because such a justification would be circular.
How does a deductive argument work?
The plot thickens: deduction

How does a deductive argument work?

**PREMISES** $A_1, \ldots A_n$

**CONCLUSION** $B$
How does a deductive argument work?

**PREMISES** $A_1, \ldots A_n$

**CONCLUSION** $B$

The plot thickens: deduction

*Inductive justification* then the justification is too weak: we only show that truth of premises often correlate with truth of conclusion.

*Deductive justification* then the justification is circular: we assume we can employ a deductive argument.
How does a deductive argument work?

**Premises** $A_1, \ldots, A_n$

**Conclusion** $B$

**Deduction**

**Inductive Justification**

**Deductive Justification**

*If* **Inductive justification** then the justification is too weak: we only show that truth of premises often correlate with truth of conclusion.
The plot thickens: deduction

How does a deductive argument work?

**Premises** \( A_1, \ldots, A_n \)

**Conclusion** \( B \)

**IF** Inductive justification then the justification is too weak: we only show that truth of premises often correlate with truth of conclusion.

**IF** Deductive justification then the justification is circular: we assume we can employ a deductive argument.
That's all Folks!