07
CHECKING VALIDITY WITH TREES (I)

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Formal Logic
Logical systems  A description of a logical system comes in three parts:

(A) Grammar: A description of what counts as a formula.

(B) Semantics: A definition of truth on an evaluation (or truth in a model); and derivatively validity and related concepts.

(C) Proofs: A description of what counts as a proof.

We have completed (A) and (B) for the logical system propositional logic. We have a description of what counts as a formula of propositional logic, and we have a definition of truth on an evaluation for formulas.

Today we move to (C) – we will give a description what counts as a proof in propositional logic. Once that's done we have given a full description of propositional logic, the target of the first part of this course.
The story so far...

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A quick reminder of some central concepts:

Logic is the study of arguments (of the kind used in philosophy), and in particular, what makes such arguments good.

Validity: generic
An argument \( \langle X, A \rangle \) is valid iff in every case, if all the premises \( X \) are true then the conclusion \( A \) is also true.

Validity: propositional logic
An argument \( \langle X, A \rangle \) is valid in propositional logic \( (X \models A) \) iff on every evaluation, if all the premises \( X \) are true then \( A \) is also true.

We’ve learned how to check whether arguments are valid using the truth table method. But that method becomes very cumbersome once we are dealing with several atomic formulas.
There are many proof systems for propositional logic. In this course we will learn one we call the tree method.
The tree method

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There is a proof for \langle X, A \rangle (X \vdash A) iff there is a closed tree for \langle X, \sim A \rangle.
There are many proof systems for propositional logic. In this course we will learn one we call the **tree method**.

There is a *proof* for \( \langle X, A \rangle \ \langle X \vdash A \rangle \) iff there is a **closed tree** for \( \langle X, \sim A \rangle \).

Intuitively, think of the tree method as a way to check for inconsistency. If \( \langle X, A \rangle \) is a valid argument then the set consisting of all the premises and the negation of the conclusion \( \{X_1, \ldots X_n, \sim A\} \) is inconsistent. The tree method is designed to produce a closed trees for inconsistent sets.
The tree method

- There are many proof systems for propositional logic. In this course we will learn one we call the tree method.

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**Q** Why does validity of \( \langle X, A \rangle \) imply inconsistency of \( \{X_1, \ldots, X_n, \sim A\} \)?

**A** Because, given the definition **validity**: **propositional logic**, if \( \langle X, A \rangle \) is valid then on every evaluation \( X_1, \ldots, X_n \) are true, \( A \) is also true. There is then no evaluation where \( X_1, \ldots, X_n \) are true, but \( A \) is false. There is then no evaluation where every member of the set \( \{X_1, \ldots, X_n, \sim A\} \) are true. And **then** \( \{X_1, \ldots, X_n, \sim A\} \) is inconsistent.
To construct trees we need rules to resolve formulas.

A tree is **closed** if all its branches are closed, and **open** otherwise.

A branch is (a) **closed** if a formula \( A \) and a formula \( \sim A \) (its negation) occur in that branch, (b) **open** iff it is complete and not closed.

**Tree rule: closure**

\[
\begin{align*}
A \\
\vdots \\
\sim A \\
\downarrow \\
X
\end{align*}
\]
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<table>
<thead>
<tr>
<th>( p )</th>
<th>( \sim )</th>
<th>( \sim )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Tree rules: conjunction and disjunction

- We add the following rules to our tree method:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$\sim (A &amp; B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1 0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1 0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1 0</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0 1</td>
</tr>
</tbody>
</table>

Tree rules: conjunction

$(A \& B) \quad \sim (A \& B)$

$\quad \sim A \quad \sim B$

Tree rules: disjunction

$(A \lor B)$
We add the following rules to our tree method:

### Tree rules: conjunction

\[ (A \land B) \]

\[ \sim (A \land B) \]

\[ \sim A \]

\[ \sim B \]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>\sim (A \land B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1 0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1 0</td>
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<tr>
<td>1</td>
<td>0</td>
<td>1 0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0 1</td>
</tr>
</tbody>
</table>

### Tree rules: disjunction

\[ (A \lor B) \]

\[ \sim (A \lor B) \]

\[ \sim A \]

\[ \sim B \]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>\sim (A \lor B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1 0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0 1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0 1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>0 1</td>
</tr>
</tbody>
</table>
Closing a tree: example 1

\(? \ \sim p \ \& \ \sim q \vdash \sim (p \ \lor \ q)\)
Closing a tree: example 1

\[ \sim p & \sim q \vdash \sim (p \lor q) \]

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 1

? \( \sim p \land \sim q \vdash \sim (p \lor q) \)

\[
\begin{array}{c}
\sim \sim A \\
\quad \mid A
\end{array}
\]

\[
\begin{array}{c}
(A \land B) \\
\quad \mid A \\
\quad \mid B
\end{array}
\]

\[
\begin{array}{c}
(A \lor B) \\
\quad \mid A \\
\quad \mid B
\end{array}
\]

\[
\sim p \land \sim q \\
\sim \sim (p \lor q)
\]
Closing a tree: example 1

? \( \sim p \& \sim q \vdash \sim (p \lor q) \)

\( \sim \sim A \)

\( \sim \sim A \)

\( (A \& B) \)

\( (A \lor B) \)

\( \sim p \& \sim q \)

\( \checkmark \sim \sim (p \lor q) \)

\( \checkmark \sim \sim (p \lor q) \)

\( \sim \sim (p \lor q) \)

\( p \lor q \)
Closing a tree: example 1

\[ \sim p \land \sim q \vdash \sim (p \lor q) \]

\( \sim \sim A \)

\( A \)

\( (A \land B) \)

\( A \)

\( B \)

\( \sim p \land \sim q \)

\( \checkmark \sim \sim (p \lor q) \)

\( p \lor q \)
Closing a tree: example 1

\( \sim p \& \sim q \vdash \sim(p \lor q) \)

\( \sim \sim A \)
\[
\begin{array}{c}
A \\
\end{array}
\]

\( (A \& B) \)
\[
\begin{array}{c}
A \\
B
\end{array}
\]

\( \checkmark \sim p \& \sim q \)
\( \checkmark \sim \sim (p \lor q) \)
\[
\begin{array}{c}
p \lor q \\
\sim p \\
\sim q
\end{array}
\]
Closing a tree: example 1

\[ \therefore \sim p \land \sim q \vdash \sim(p \lor q) \]

\[ \begin{array}{c}
\sim \sim A \\
\sim A \\
\hline \\
\sim \sim(p \lor q)
\end{array} \]

\[ \begin{array}{c}
A \\
B \\
\hline \\
\sim \sim((A \land B)) \\
\sim (A \land B) \\
\hline \\
\sim p \\
\sim q
\end{array} \]

\[ \begin{array}{c}
\sim \sim ((A \lor B)) \\
\sim (A \lor B) \\
\hline \\
\sim p \\
\sim q
\end{array} \]
Closing a tree: example 1

? \( \sim p \& \sim q \vdash \sim(p \lor q) \)

\( \sim \sim A \)
  \( \quad \)
  \( A \)

\( (A \& B) \)
  \( \quad \)
  \( A \)
  \( \quad \)
  \( B \)

\( (A \lor B) \)
  \( \quad \)
  \( A \)
  \( \quad \)
  \( B \)

✓ \( \sim p \& \sim q \)

✓ \( \sim \sim (p \lor q) \)

✓ \( p \lor q \)

✓ \( \sim p \)

✓ \( \sim q \)

✓ \( p \)

✓ \( q \)

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 1

\( \sim p \& \sim q \vdash \sim(p \lor q) \)

\( \sim\sim A \\
\quad A \\
\)

\( (A \& B) \\
\quad A \\
\quad B \\
\)

\( (A \lor B) \\
\quad A \\
\quad B \\
\)

\( \checkmark \sim p \& \sim q \\
\checkmark \sim\sim(p \lor q) \\
\checkmark p \lor q \\
\sim p \\
\sim q \\
\quad p \\
\quad q 

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 1

\[
\begin{align*}
\checkmark \quad \neg p & \land \neg q \vdash \neg (p \lor q) \\
\neg \neg A \\
\checkmark \quad \neg p \land \neg q \\
\checkmark \quad \neg \neg (p \lor q) \\
\checkmark \quad p \lor q \\
\checkmark \quad \neg p \\
\checkmark \quad \neg q \\
\checkmark \quad p \\
\checkmark \quad q \\
\checkmark \quad \times \\
\checkmark \quad \times \\
\checkmark \quad \times \\
\end{align*}
\]

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 2

\[ \sim (p \lor q) \vdash \sim p \& \sim q \]
Closing a tree: example 2

\(? \quad \neg(p \lor q) \vdash \neg p \land \neg q\)

\[\neg(A \lor B)
\frac{}{\quad \neg A}
\frac{\quad \neg B}{\neg(A \land B)}\]

\[\neg(p \lor q)
\frac{}{\quad \neg(\neg p \land \neg q)}\]
Closing a tree: example 2

? \sim(p \lor q) \vdash \sim p & \sim q

\sim(A \lor B)
  \sim A
  \sim B

\sim(A \land B)
  \sim A
  \sim B

\sim(p \lor q)
\sim(\sim p & \sim q)
Closing a tree: example 2

\(? \sim (p \lor q) \vdash \sim p & \sim q\)

\(\sim (A \lor B)\)
\(\quad \sim A\)
\(\quad \sim B\)

\(\sim (A \land B)\)
\(\quad \sim A\quad \sim B\)

\(\checkmark \sim (p \lor q)\)
\(\quad \sim (\sim p & \sim q)\)
\(\quad \sim p\)
\(\quad \sim q\)
? \((p \lor q) \vdash \neg p \land \neg q\)

\[-(A \lor B)\]
\[-A\]
\[-B\]

\[-(A \land B)\]
\[-A\]
\[-B\]

\[\checkmark -(p \lor q)\]
\[-(\neg p \land \neg q)\]
\[-\neg p\]
\[-\neg q\]
Closing a tree: example 2

\[ \sim(p \lor q) \vdash \sim p \land \sim q \]

\[ \sim A \]
\[ \sim B \]

\[ \sim(A \lor B) \]

\[ \vdash \sim p \land \sim q \]

\[ \sim \sim p \]
\[ \sim \sim q \]
Closing a tree: example 2

\[ \sim(p \lor q) \vdash \sim p \land \sim q \]

\[ \sim(A \lor B) \]
\[ \quad \sim A \]
\[ \quad \sim B \]

\[ \sim(A \land B) \]
\[ \quad \sim A \quad \sim B \]

\[ \checkmark \sim(p \lor q) \]
\[ \checkmark \sim(\sim p \land \sim q) \]
\[ \quad \sim p \]
\[ \quad \sim q \]
\[ \quad \sim \sim p \quad \sim \sim q \]

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 2

✓ \( \sim(p \lor q) \vdash \sim p \land \sim q \)

\[
\begin{array}{c}
\sim(A \lor B) \\
\mid \\
\sim A \\
\sim B
\end{array}
\]

\[
\begin{array}{c}
\sim(A \land B) \\
\sim A \\
\sim B
\end{array}
\]

✓ \( \sim(p \lor q) \)

✓ \( \sim(\sim p \land \sim q) \)

\[
\begin{array}{c}
\sim p \\
\sim q
\end{array}
\]

\[
\begin{array}{c}
\sim p \\
\times
\end{array}
\]

\[
\begin{array}{c}
\sim q \\
\times
\end{array}
\]

All branches (i.e. the tree) close(s); there is a tree proof.
Let’s add some new rules to our tree method:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$\sim (A \supset B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
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</tbody>
</table>

Tree rules: conditional

$$(A \supset B) \quad \sim (A \supset B)$$

$\sim A \quad B$

Tree rules: biconditional

$$(A \equiv B) \quad \sim (A \equiv B)$$

$\sim A \quad \sim B$$
Let’s add some new rules to our tree method:

**Tree rules: conditional**

\[(A \supset B) \quad \sim(A \supset B)\]

\[\sim A \quad B\]

\[\sim B\]

**Tree rules: biconditional**

\[(A \equiv B) \quad \sim(A \equiv B)\]

\[A \quad \sim A \quad \sim A \quad A\]

\[B \quad \sim B \quad B \quad \sim B\]
Closing a tree: example 3

\[ p \supset q, \sim p \supset \sim q \vdash p \equiv q \]
Closing a tree: example 3

\(? \quad p \supset q, \sim p \supset \sim q \vdash p \equiv q\)

\[
\neg(A \equiv B) \\
\quad \neg A \\
\quad A \\
\quad B \\
\quad \neg B
\]

\[
(A \supset B) \\
\quad \neg A \\
\quad B
\]

\[
\neg(p \equiv q) \\
\quad \neg p \\
\quad \neg q
\]

\[
p \supset q \\
\sim p \supset \sim q
\]
Closing a tree: example 3

\[ p \supset q, \sim p \supset \sim q \vdash p \equiv q \]

\[ \sim (A \equiv B) \]
\[ \sim A \quad A \]
\[ B \quad \sim B \]

\[ (A \supset B) \]
\[ \sim A \quad B \]

\[ p \supset q \]
\[ \sim p \supset \sim q \]
\[ \sim (p \equiv q) \]
Closing a tree: example 3

\[ ? \quad p \supset q, \sim p \supset \sim q \vdash p \equiv q \]

\[ \sim(A \equiv B) \]
\[ \sim A \quad A \]
\[ B \quad \sim B \]

\[ (A \supset B) \]
\[ \sim A \quad B \]

\[ p \supset q \]
\[ \sim p \supset \sim q \]
\[ \checkmark \sim(p \equiv q) \]
\[ \sim p \quad p \]
\[ q \quad \sim q \]

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 3

\(? \ p \supset q, \ \sim p \supset \sim q \vdash p \equiv q\)

\[
\begin{array}{c}
\sim(A \equiv B) \\
\sim A & A \\
B & \sim B \\
\end{array}
\]

\[
\begin{array}{c}
(A \supset B) \\
\sim A & B \\
\end{array}
\]

\[
\begin{array}{c}
p \supset q \\
\sim p \supset \sim q \\
\checkmark \sim(p \equiv q) \\
\sim p & p \\
q & \sim q \\
\end{array}
\]
Closing a tree: example 3

\[ ? \quad p \supset q, \sim p \supset \sim q \vdash p \equiv q \]

\[ \sim (A \equiv B) \]
\[ \sim A \quad A \]
\[ B \quad \sim B \]

\[ (A \supset B) \]
\[ \sim A \quad B \]

\[ p \supset q \]
\[ \checkmark \sim p \supset \sim q \]
\[ \checkmark \sim (p \equiv q) \]

\[ \sim p \quad p \]
\[ \sim q \quad \sim p \]
\[ \sim q \quad \sim q \]

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 3

\[ p \supset q, \sim p \supset \sim q \vdash p \equiv q \]

\[ \sim (A \equiv B) \]
\[ \sim A \quad A \]
\[ B \quad \sim B \]

\[ (A \supset B) \]
\[ \sim A \quad B \]

\[ p \supset q \]
\[ \checkmark \sim p \supset \sim q \]
\[ \checkmark \sim (p \equiv q) \]

\[ \sim p \]
\[ q \]
\[ \checkmark \sim p \sim q \]

\[ p \]
\[ \sim q \]
\[ \sim \sim p \]
\[ \sim q \]

\[ \sim \sim p \]
\[ \sim q \]

\[ \sim \sim p \]
\[ \sim q \]
Closing a tree: example 3

? $p \supset q, \sim p \supset \sim q \vdash p \equiv q$

$\sim (A \equiv B)$
- $\sim A$
- $A$
- $B$
- $\sim B$

$(A \supset B)$
- $\sim A$
- $B$

$\vdash p \supset q$
- $\checkmark \sim p \supset \sim q$
- $\checkmark \sim (p \equiv q)$

$\sim p$
- $\sim q$
- $\sim \sim p$
- $\sim q$
- $\times$
- $\times$
? $p \supset q$, $\sim p \supset \sim q \vdash p \equiv q$

\[\sim (A \equiv B)\]
\[\sim A \quad A\]
\[B \quad \sim B\]

\[(A \supset B)\]
\[\sim A \quad B\]

\[p \supset q\]
\[\checkmark \sim p \supset \sim q\]
\[\checkmark \sim (p \equiv q)\]

\[\sim p \quad p\]
\[q \quad \sim q\]

\[\sim \sim p \quad \sim q\]
\[\times \quad \times\]

\[\sim \sim p \quad \sim q\]
\[\times \quad \times\]
Closing a tree: example 3

? \( p \supset q, \neg p \supset \neg q \vdash p \equiv q \)

\( \neg(A \equiv B) \)
- \( \neg A \)
- \( A \)
- \( B \)
- \( \neg B \)

\( (A \supset B) \)
- \( \neg A \)
- \( B \)

\( \checkmark p \supset q \)
\( \checkmark \neg p \supset \neg q \)
\( \checkmark \neg(p \equiv q) \)

\( \neg \neg p \)
- \( \neg q \)

- \( \neg \neg p \)
  - \( \neg q \)
    - \( \neg p \)
    - \( q \)
  - \( \neg p \)
    - \( q \)
    - \( \neg p \)
    - \( q \)

All branches (i.e. the tree) close(s); there is a tree proof.
Closing a tree: example 3

\[ p \supset q, \sim p \supset \sim q \vdash p \equiv q \]

\[ \sim (A \equiv B) \]
\[ \sim A \quad A \]
\[ \sim B \quad \sim B \]

\[ (A \supset B) \]
\[ \sim A \quad B \]

\[ \checkmark p \supset q \]
\[ \checkmark \sim p \supset \sim q \]
\[ \checkmark \sim (p \equiv q) \]

\[ \sim p \quad q \quad \sim q \]
\[ \sim \sim p \quad \sim q \quad \sim \sim p \quad \sim q \]
\[ \times \quad \times \quad \sim p \quad q \quad \sim p \quad q \]
Closing a tree: example 3

✓ \ p \supset q, \ \sim p \supset \sim q \vdash p \equiv q

\sim(A \equiv B)
\sim A \quad A
B \quad \sim B

(A \supset B)
\sim A \quad B

\checkmark \ p \supset q
\checkmark \sim p \supset \sim q
\checkmark \sim(p \equiv q)

\sim p
\sim q

\sim \sim p \quad \sim q
\checkmark \sim p \quad \sim q
\sim \sim p \quad \sim q
\checkmark \sim p \quad \sim q

\checkmark \sim p \quad \sim q
\checkmark \sim p \quad \sim q

All branches (i.e. the tree) close(s); there is a tree proof.
We can give a summary of how to use the tree method:

**HOW TO: CHECK WHETHER THERE IS A TREE PROOF**

(a) Where $\langle X, A \rangle$ is an argument, write down all the premises $X$ and negation of conclusion $\sim A$ (the root).

(b) Then apply the rules for the connectives to produce branches on the tree. Each time you apply a rule on a formula, remember to mark that formula as resolved with ✓.

(c) If a formula along with its negation occurs in a branch of the tree, then mark that branch as closed with ×.

There are then two possibilities:

(d) If you thereby produce a tree where all branches are closed, you have shown that there is a tree proof for $\langle X, A \rangle$ ($X \vdash A$).

(e) If you thereby produce a tree where all the formulas are resolved and at least one branch is open, you have shown that there isn’t a tree proof for $\langle X, A \rangle$ ($X \nvdash A$).
Closing a tree: example 4

? \( p \supset (q \lor r) \), \( \neg (p \& r) \vdash p \supset q \)
Closing a tree: example 4

\(? \quad p \supset (q \lor r), \sim(p \land r) \vdash p \supset q\)

\[\sim (A \supset B) \quad (A \supset B)\]
\[\begin{array}{c}
A \\
\sim B
\end{array} \quad \begin{array}{c}
\sim A \\
B
\end{array}\]

\[(A \lor B)\]
\[\begin{array}{c}
A \\
B
\end{array}\]

\[\sim (A \land B)\]
\[\begin{array}{c}
\sim A \\
\sim B
\end{array}\]

\[p \supset (q \lor r)\]
\[\sim (p \land r)\]
\[\sim (p \supset q)\]
Closing a tree: example 4

\[ p \supset (q \lor r), \neg(p \land r) \vdash p \supset q \]

\[ \neg(A \supset B) \]
\[ \neg\neg A \supset B \]
\[ A \]
\[ \neg B \]

\[ (A \lor B) \]
\[ A \]
\[ B \]

\[ \neg(A \land B) \]
\[ \neg A \]
\[ \neg B \]

\[ p \supset (q \lor r) \]
\[ \neg(p \land r) \]
\[ \neg(p \supset q) \]
Closing a tree: example 4

\[ p \supset (q \lor r), \sim (p \& r) \vdash p \supset q \]

\[
\begin{array}{c}
\neg (A \supset B) & (A \supset B) \\
\longleftarrow A & \neg A \quad B \\
\neg B & \\
\end{array}
\]

\[
\begin{array}{c}
(A \lor B) \\
\quad A \quad B \\
\end{array}
\]

\[
\begin{array}{c}
\neg (A \& B) \\
\longleftarrow \neg A \quad \neg B \\
\end{array}
\]

\[
\begin{array}{c}
p \supset (q \lor r) \\
\neg (p \& r) \\
\checkmark \neg (p \supset q) \\
\longleftarrow p \\
\neg q
\end{array}
\]
Closing a tree: example 4

\[ p \supset (q \lor r), \sim(p \land r) \vdash p \supset q \]

\[
\begin{array}{c}
\sim(A \supset B) \\
\ \ \\
\sim B \\
\end{array}
\quad
\begin{array}{c}
(A \supset B) \\
\ \ \\
\sim A \\
\sim B
\end{array}
\]

\[
\begin{array}{c}
(A \lor B) \\
\ \ \\
A \\
B
\end{array}
\]

\[
\begin{array}{c}
\sim(A \land B) \\
\ \ \\
\sim A \\
\sim B
\end{array}
\]

\[
\begin{array}{c}
p \supset (q \lor r) \\
\sim(p \land r) \\
\checkmark \sim(p \supset q) \\
p \\
\sim q
\end{array}
\]
Closing a tree: example 4

\[ p \supset (q \lor r) \land \neg(p \land r) \vdash p \supset q \]

\[
\begin{align*}
\neg(A \supset B) & \quad (A \supset B) \\
\quad \quad A & \quad \sim A \quad B \\
\quad \quad \sim B & \\
\end{align*}
\]

\[
\begin{align*}
(A \lor B) & \\
\quad \quad A & \quad B \\
\end{align*}
\]

\[
\begin{align*}
\neg(A \land B) & \\
\quad \quad \sim A & \quad \sim B \\
\end{align*}
\]

\[
\begin{align*}
\checkmark p \supset (q \lor r) & \\
\neg(p \land r) & \\
\checkmark \neg(p \supset q) & \\
\quad \quad p & \\
\quad \quad \sim q & \\
\quad \quad \sim p & \quad q \lor r \\
\end{align*}
\]
Closing a tree: example 4

? \( p \supset (q \lor r), \sim(p \& r) \vdash p \supset q \)

\[
\begin{align*}
\sim(A \supset B) & \quad (A \supset B) \\
\quad A & \quad \sim A \quad B \\
\quad \sim B & \\
\end{align*}
\]

\[
\begin{align*}
(A \lor B) & \\
\quad A & \quad B \\
\end{align*}
\]

\[
\begin{align*}
\sim(A \& B) & \\
\quad \sim A & \quad \sim B \\
\end{align*}
\]

\[
\begin{align*}
\checkmark p \supset (q \lor r) \\
\sim(p \& r) \\
\checkmark \sim(p \supset q) \\
\sim q \\
\sim p & \quad q \lor r \\
\times &
\end{align*}
\]
Closing a tree: example 4

\[ p \supset (q \lor r), \neg (p \land r) \vdash p \supset q \]

\[
\begin{align*}
\neg (A \supset B) & \quad (A \supset B) \\
\quad A & \quad \neg A \quad B \\
\neg B & \\
\end{align*}
\]

\[
\begin{align*}
(A \lor B) & \\
\quad A & \quad B \\
\end{align*}
\]

\[
\begin{align*}
\neg (A \land B) & \\
\quad \neg A & \quad \neg B \\
\end{align*}
\]

\[
\begin{align*}
\checkmark p \supset (q \lor r) & \\
\neg (p \land r) & \\
\checkmark \neg (p \supset q) & \\
\quad p & \\
\quad \neg q & \\
\quad \neg p & \quad q \lor r \\
\quad \times & \\
\end{align*}
\]
Closing a tree: example 4

? \( p \supset (q \lor r), \sim (p \& r) \vdash p \supset q \)

\[
\begin{align*}
\neg (A \supset B) & \quad (A \supset B) \\
\quad & \\
\quad A & \quad \neg A \quad B \\
\quad & \\
\quad & \sim B \\
\end{align*}
\]

\[
\begin{align*}
(A \lor B) & \quad (A \lor B) \\
\quad & \\
\quad A & \quad B \\
\end{align*}
\]

\[
\begin{align*}
\neg (A \& B) & \quad (A \& B) \\
\quad & \\
\quad \neg A & \quad \neg B \\
\quad & \\
\quad & \\
\end{align*}
\]
Closing a tree: example 4

\( \vdash p \supset (q \lor r), \sim(p \land r) \vdash p \supset q \)
Closing a tree: example 4

\[ p \supset (q \lor r), \sim(p \land r) \vdash p \supset q \]

\[ \sim(A \supset B) \quad (A \supset B) \]
\[ \quad \quad \quad \quad A \quad \sim A \quad B \]
\[ \sim B \]

\[ (A \lor B) \]
\[ \quad \quad \quad A \quad B \]

\[ \sim(A \land B) \]
\[ \quad \quad \quad \sim A \quad \sim B \]

\[ \checkmark p \supset (q \lor r) \]
\[ \sim(p \land r) \]
\[ \checkmark \sim(p \supset q) \]
\[ \checkmark \sim q \]
\[ \sim p \quad \checkmark q \lor r \]
\[ \times \quad q \quad r \]
\[ \times \]
Closing a tree: example 4

\[ p \supset (q \lor r), \sim (p \land r) \vdash p \supset q \]

\[
\sim (A \supset B) \quad (A \supset B) \\
\quad A \quad \sim A \quad B \\
\quad \sim B
\]

\[
(A \lor B) \\
A \quad B
\]

\[
\sim (A \land B) \\
\sim A \quad \sim B
\]

\[
\checkmark p \supset (q \lor r) \\
\checkmark \sim (p \land r) \\
\checkmark \sim (p \supset q) \\
\]

\[
\sim p \quad \checkmark q \lor r \\
\checkmark \sim q \\
\quad \sim p \quad \sim r
\]
Closing a tree: example 4

\( p \supset (q \lor r), \neg (p \& r) \vdash p \supset q \)

\[
\begin{align*}
\neg (A \supset B) & \quad (A \supset B) \\
\quad & \quad A \quad \neg A \\ 
\quad & \quad B \\
\end{align*}
\]

\[
\begin{align*}
(A \lor B) & \\
\quad & \quad A \\ 
\quad & \quad B \\
\end{align*}
\]

\[
\begin{align*}
\neg (A \& B) & \\
\quad & \quad \neg A \\ 
\quad & \quad \neg B \\
\end{align*}
\]

\[
\begin{align*}
\checkmark p \supset (q \lor r) & \\
\checkmark \neg (p \& r) & \\
\checkmark \neg (p \supset q) \\
\end{align*}
\]

\[
\begin{align*}
\checkmark \neg p & \\
\checkmark q \lor r & \\
\checkmark \neg p & \\
\checkmark \neg r & \\
\end{align*}
\]
Closing a tree: example 4

\[ p \supset (q \lor r), \neg(p \land r) \vdash p \supset q \]
Summary of the tree method

There is a proof for \( \langle X, A \rangle \ (X \vdash A) \) iff there is a closed tree for \( \langle X, \sim A \rangle \).

A tree is closed if all its branches are closed, and open otherwise.

A branch is (a) closed if a formula \( A \) and a formula \( \sim A \) (its negation) occur in that branch, (b) open iff it is complete and not closed.

- A tree is complete iff all its branches are complete or closed. A branch is complete iff all formulas occurring in it are resolved. All the atomic and negated atomic formulas count as resolved. A formula is resolved (marked with ✓) when a rule has been applied to it.