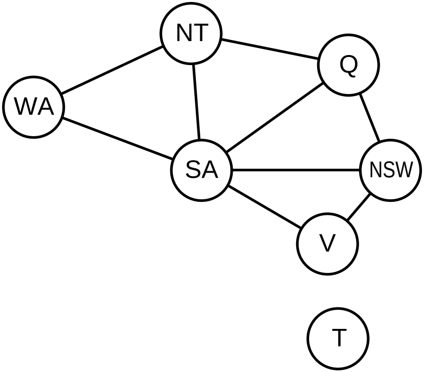
Nodes from a goal set

$$G_n \approx \{\text{nodes with distance } n \text{ from } G\}$$



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$\{\mathrm{V}\}_0 = \{\mathrm{V}\}$

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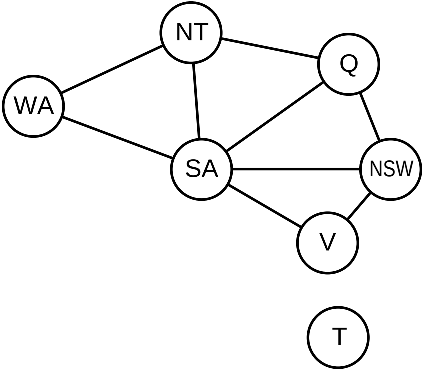

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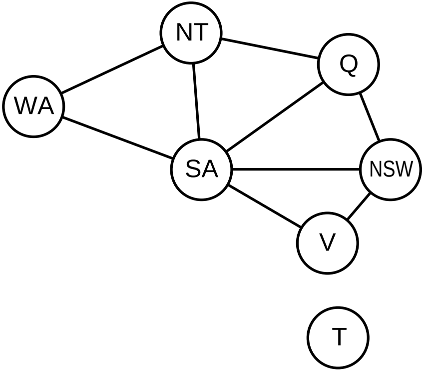

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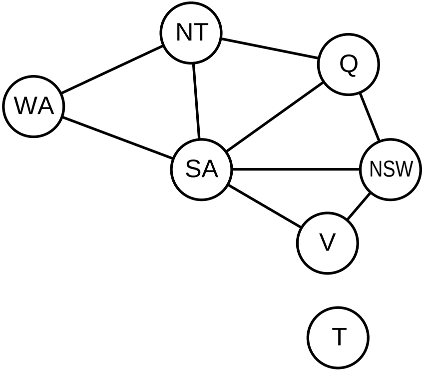

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Distance d_G to minimize

$$d_G(s) \ := \begin{cases} n & \text{if } s \in G_n \\ \infty & \text{otherwise} \end{cases}$$

Refine

$$\delta_G(s) \ := \begin{cases} 1 & \text{if } s \in G \\ 0 & \text{otherwise} \end{cases}$$

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$$r_G(s) \ = \frac{1}{2} r_G(s') \text{ if } arc(s, s') \text{ and } d_G(s') < d_G(s).$$

Rewards looking ahead

$$\begin{aligned} H_0(s) &:= \delta_G(s) \\ H_{n+1}(s) &:= \delta_G(s) + \frac{1}{2}\max\{H_n(s') \mid arc_=(s,s')\} \end{aligned}$$

where $arc_=$ encodes move/rest

$$arc_=(s,s') \iff arc(s,s') \text{ or } s = s'.$$

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For $s \in G_0$,

$$H_{n+1}(s) = 1 + \frac{1}{2}H_n(s) = a_{n+1}$$

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$$a_n := \sum_{k=0}^{n} 2^{-k} = 2(1 - 2^{-(n+1)})$$

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Rewards looking ahead: $\lim_{n\to\infty} H_n(s) = 2r_G(s)$

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$H = \lim_{n\to\infty} H_n$

$$H(s) \ = \delta_G(s) + \frac{1}{2}\max\{H(s') \mid arc_=(s,s')\}$$

a foolproof heuristic for the shortest solution

`Frontier = [Hd|Tl]` with $H(\texttt{Hd}) \ \geq H(s)$ for all s in `Tl`.

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What if arcs have different costs?

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a foolproof heuristic for the shortest solution

`Frontier = [Hd|Tl]` with $H(\texttt{Hd}) \; \geq H(s)$ for all s in `Tl`.

What if arcs have different costs?

Extend $\delta_G(s)$ to arc/rest s, s'

$$Q_0(s,s') \; := \; \begin{cases} 1 & \text{if } s = s' \in G \\ -\text{cost}(s,s') & \text{else if } arc(s,s') \\ -\max_{s_1,s_2}\text{cost}(s_1,s_2) & \text{otherwise} \end{cases}$$

and $H_{n+1}(s)$ to

$$Q_{n+1}(s,s') \; := \; Q_0(s,s') + \frac{1}{2}\max\{Q_n(s',s'') \mid arc_=(s',s'')\}$$

Discounted rewards ($0 \leq \gamma < 1$)

(immediate) rewards $r_1, r_2, r_3, \ldots$ at times $1, 2, 3, \ldots$ give a γ-discounted value of

$$V \ := r_1 + \gamma r_2 + \gamma^2 r_3 + \cdots \ = \ \sum_{i \geq 1} \gamma^{i-1} r_i$$

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where V_t is the value from time step t on ($V_1 = V$)

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which is bound by bounds on r_i

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$Q = \lim_{n\to\infty} Q_n$

$$Q(s,s') := Q_0(s,s') + \frac{1}{2}\max\{Q(s',s'') \mid arc_=(s',s'')\}$$

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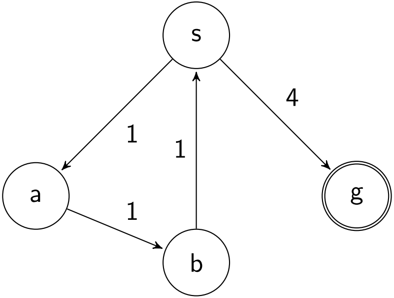

soln not chosen

Q(s,g) = -3

Q(s,a) = -2 = Q(a,b) = Q(b,s)

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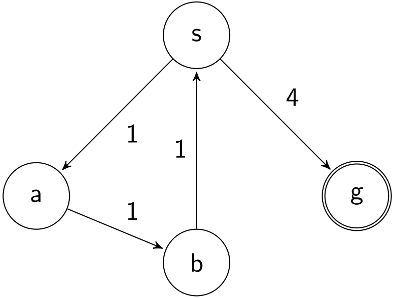

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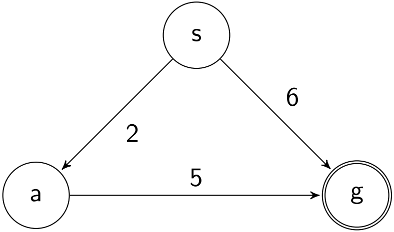

costlier soln chosen
Q(s,g) = -5
Q(a,g) = -4 = Q(s,a)

Raising the reward

Adjust $Q_0(s, s')$ to

$$R(s, s') \;:=\; \begin{cases} r & \text{if } s = s' \in G \\ -\text{cost}(s, s') & \text{else if } arc(s, s') \end{cases}$$

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for solutions up to $n - 1$ arcs with arcs costing $\leq c$

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Let

$$V(s) \;:=\; \max\{Q(s, s') \mid arc_=(s, s')\}$$

so for $0 \leq i < n$, $s' \in G_i$ and $arc(s, s')$,

$$V(s') \;\geq 2^{n-i}(n - i)c$$

$$V(s) \;\geq\; -c + \frac{1}{2}V(s') \;\geq\; 2^{n-(i+1)}(n - (i+1))c \;\geq\; 2c.$$

Recap

From node s, find path to goal via s' maximizing

$$Q(s,s') \;:=\; R(s,s') + \frac{1}{2}V(s')$$

with discount $\frac{1}{2}$ on future $V(s')$, contra

$$\begin{aligned}\text{cost}(s_1 \cdots s_k) \;&=\; \sum_{i=1}^{k-1} \text{cost}(s_i, s_{i+1}) \\ &\neq\; \sum_{i=1}^{k-1} 2^{1-i}\text{cost}(s_i, s_{i+1})\end{aligned}$$

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NEXT: more uncertainty, approached via approximations like

$$\begin{aligned} Q_n(s,s') \ &\approx \ Q(s,s') \text{ up to look ahead } n \\ Q(s,s') \ &= \ \lim_{n\to\infty} Q_n(s,s'). \end{aligned}$$