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Irish Logarithms Part 2

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Logarithms

The seasoned slide rule collector will now ask: why didn't they use logarithms for direct-multiplication-by-addition? There is a mechanical reason. The smallest distance between the logarithms of the simple products is log(64)−log(63) = 0.00684. This means that on a scale of log(81) = 1.909 a probing precision of 0.00684 must be reached, so 0.4%, which is even smaller than the 1/81 = 1.2% required for the one-hole multiplication table previously discussed.

However, number theory provides an alternative: the **Jacobi index** of an integer.[1]

Take a prime number p. The Jacobi index $\mathrm{Ind}(z_i)$ of an integer z_i between 1 and $p-1$ is an integer between 0 and $p-2$ with the property

$$\mathrm{Ind}(z_1 \times z_2) = \mathrm{Ind}(z_1) + \mathrm{Ind}(z_2)$$

For any number z_i between 1 and $p-1$ a unique Jacobi index can be determined.

So using Ind(), we can multiply by adding. Appendix 1 presents in detail the theory and construction of the indices.

For an improved one-hole Verea-type calculator we only need to determine the indices of the 36 unique numbers that appear as a simple product. We will only show the indices of the **primes** less than 10, i.e. 1, 2, 3, 5 and 7. The indices of the remaining 31 unique numbers follow easily from the indices of the primes.

The following table shows the indices of the primes for the first two sets that we find using the construction method of Appendix 1:

z	1	2	3	5	7
Ind(z)	0	1	18	44	7

Ind(*z*)	0	1	8	44	27

In both solutions largest index is 88 (for 5 × 5), which would require a deeper hole than the 81 in the primitive one-hole multiplication table.

In 1909, the Irish accountant Percy Ludgate (1883–1922) proposed a calculator which contained a kind of index.[2] He determined these indexes without the theoretical basis given above. His indices of prime numbers are:

z	1	2	3	5	7
Ind(*z*)	0	1	7	23	33

The largest index is **66** (for 7 × 7), so that's an improvement. Ludgate did not use the method of Appendix 1. In fact, I failed to reconstruct his indices by the method of Appendix 1. That does not mean that it is impossible, but I'm not sure that (number-theoretically) this method is required for the indices of the restricted set of 36 simple products.

Ludgate probably got the indices as follows: you obviously start with Ind(1) = 0, then take Ind(2) = 1, so Ind(4) = 2, Ind(8) = 3, ... Ind(64) = 6. The next unused index is 7. Let's choose this as Ind(3). So Ind(3) = 7, Ind(2×3) = 8, ... Ind(3×16) = 11 (we do not go beyond *z* = 81). The index values 14, 21 and 28 = Ind(81) are now occupied as well. Ludgate will have obtained the values for Ind(5) and Ind(7) by trial and error. He writes that he found them “after some difficulty”.[2] In the English literature, these indices are called “Irish logarithms”.

In 1913, K. Hoecken published an extensive article[1] on multiplicating machines which does refer to the number-theoretical basis. Hoecken ultimately ends up with the same table as Ludgate and admits that this table is purely empirical. In a footnote, added at the time of printing, he mentions two other sets of empirical indices:

z	1	2	3	5	7	max Index	Author
Ind(*z*)	0	1	13	21	30	**60**	Remak
Ind(*z*)	0	8	13	1	30	**60**	A. Korn

These solutions are slightly better. The solution of Korn differs from all others because Ind(2) ≠ 1. Again, I have not been able to reconstruct these indexes using the method in Appendix 1. For Ludgate, Korn and Remake Ind(49) is the highest, with values of

66, **60** and **60**, while “my” indices, using the number-theoretical method, have a maximum of **88** at Ind(25).

Slide rule

Joh. Schumacher, Professor at the Bavarian Cadet Corps, published in 1909, so in the same year as Ludgate, a design for a slide rule with indices[3]. This ruler is manufactured by A.W. Faber as Model 366[4] and contains indices for **all** numbers between 1 and 100, not only for the 36 simple products. The table below shows only the indices for the primes <101.

z	1	2	3	5	7	11	13	17	19	23	29	31	37
Ind (*z*)	0	1	69	24	9	13	66	30	96	86	91	84	56
z	41	43	47	53	59	61	67	71	73	79	83	89	97
Ind (*z*)	45	42	58	23	29	77	81	44	61	64	89	21	52

On the slide rule, all indices are marked “modulo 100”: Ind(9) = Ind (3×3) = 69 + 69 = 138 mod 100 = 38, so at the 39th mark is at a 9 (the first mark is at Ind(1) = 0).

The user should note that the product is taken modulo 101, and the sum of the indices modulo 100:

Ind(23 × 23) = Ind(23) + Ind(23) = 86 + 86 = 172 = 72 mod 100 = Ind(24 mod 101)

(The index-table shown above only contains **prime** *z*-values. Ind(24) = Ind(2*2*2*3) = 1+1+1+69 = 72. The slide rule contains Ind(*z*) for **all** *z*., so you can immediately *z* for find Ind(*z*)=72)

The “modulo 100” is executed on the Schumacher-slide rule in the same way as a transition to a higher order on a normal slide rule. So we read off 23×23 = 24, which is correct “modulo 101”. In practice, we want to have the real product, so we have to add a few times 101 to the result. In this case we expect the final digit of the product to be 3 × 3 = 9. Therefore we add 9−4 = 5 times 101 to the result, and finally get 529. It is clear that the educational value of this slide rule is greater than the practical one.

I do not know any slide rules that only contain the “Irish Logarithms” for simple products. Their usefulness would be limited: they can only calculate products of two numbers <10. If you want to experiment with them, print the PDF versions of the slide rules according to Ludgate, Korn, Remak and my two solutions, and cut them at the lines indicated. For comparison, the Schumacher slide rule and "ordinary" slide rules with marks for simple products (a version with one decade and a version with two decades) are also available as a PDF.

Epilogue

The "Irish Logarithms" have never been used in a real calculator. You can play with them in an online animation. The applicability of the Schumacher slide rule is limited. It is surprising that A.W. Faber has taken this slide rule into production. There is even a design of a disk-shaped version of the slide rule.[5] The disk was never brought to market.[4] The world was apparently too small for two Schumacher slide rules.

Notes

1. K. Hoecken, "Die Rechenmaschinen von Pascal bis zur Gegenwart, unter besonderer Berücksichtigung der Multiplikationsmechanismen", *Sitzungsberichte Berliner Math. Gesellsch.* 13 (Feb. 1913) pages 8–29.
2. B. Randell, "Ludgate's analytical machine of 1909", *The Computer Journal*, 14^3 (1971) pages 317–326
3. Dr. Joh. Schumacher, "Ein Rechenschieber mit Teilung in gleiche Intervalle auf der Grundlage der zahlentheoretischen Indizes. Für den Unterricht konstruiert", München, 1909. See the Rechnerlexikon.
4. Dieter von Jezierski, Detlef Zerfowski, Paul Weinmann: "A.W. Faber Model 366 - System Schumacher. A Very Unusual Slide Rule", *Journal of the Oughtred Society,* 13^2 (2004) pages 10–17.
5. "Technik Geschichte: Beiträge zur Geschichte der Technik und Industrie", VDI, 1933, p. 153–154.

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Subpages (1): Jacobi indices

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