Stationarity I

Definition (Strict Stationarity)

The distribution of the random process has certain attributes that are the same everywhere. Strict stationarity indicates that for any number k of any sites $x_1, x_2, \cdots, x_k$, the joint cumulative distribution of $(s(x_1), \cdots, s(x_k))$ remains the same under an arbitrary translation h:

$$P(s(x_1), \cdots, s(x_k)) = P(s(x_1 + h), \cdots, s(x_k + h))$$

If the random process is strictly stationary, its moments if they exist are also invariant under translations.

Stationarity II

Definition (weak stationarity)

A process $s(x)$ is said second order stationary (or weakly stationary, or wide-sense stationary) when

1. the mean of the process does not depend on x: $\mathbb{E}[s(x)] = \mu$
2. the variance of the process does not depend on x: $\mathbb{E}[(s(x) - \mu)^2] = \sigma^2$
3. the covariance between $s(x)$ and $s(x + h)$ only depends on h:

$$\begin{aligned} \mathbb{C}\text{ov}\,[s(x), s(x+h))] &= \mathbb{E}\left[(s(x) - \mathbb{E}[s(x)])\;(s(x+h) - \mathbb{E}[s(x+h)])\right] \\ &= \mathbb{E}\left[(s(x) - \mu)\;(s(x+h) - \mu)\right] \\ &= C(h) \end{aligned}$$

Note that $C(0) = \sigma^2$.

Stationarity III

- For a weakly stationary process, the autocorrelation can also be defined as:

$$\rho(h) = \frac{C(h)}{C(0)}$$

  with $C(0)$ is the covariance at lag 0, i.e. σ^2.

- If a random field with the function $C(h)$ only dependent on the distance $\|h\|$ and not on its orientation, it is said to be isotropic.

Stationarity IV

Definition (Intrinsic Stationarity)

$s(x)$ is said to be an intrinsic random function such that:

$$\mathbb{E}[s(x+h) - s(x)] = 0$$

and

$$\mathbb{V}\text{ar}[s(x+h) - s(x)] = 2\,\gamma(h)$$

The function $\gamma(h)$ is called the variogram.

Stationarity V

Exercises: For second-order stationary processes,

1. Express $\gamma(h)$ w.r.t. $C(h)$.
2. Express $\gamma(h)$ w.r.t. $\rho(h)$.
3. Show that a process that is weakly stationary is also intrinsically stationary.

Brownian Motion & Ornstein-Uhlenbeck Processes

- A second order stationary process is also an intrinsic stationary process.
- But an intrinsic stationary process is not always a second order stationary process or a strictly stationary process.
- To illustrate the weakly stationary and intrinsic stationary, we look at the following processes:
  - Brownian Motion,
  - Ornstein-Uhlenbeck Process.

Brownian Motion I

Definition (diffusion & Brownian Motion)

A diffusion is a continuous time Stochastic Process $s(t)$ with the following properties:

1. $s(0) = 0$,
2. $s(t)$ has independent increments,
3. $P(s(t_2)\,|\,s(t_1))$ has a density function $f(s(t_2)|t_1, t_2, s(t_1))$.

Standard Brownian motion is a diffusion $s(t)$, $t \geq 0$ satisfying the following:

1. $s(0) = 0$.
2. $s(t)$ has independent increments.
3. For $t_2 > t_1$, $s(t_2) - s(t_1) \sim \mathcal{N}(0, \sigma^2(t_2 - t_1))$.

Brownian Motion II

- $s(t)$ has independent increments means that for all times $0 \leq t_1 \leq t_2 \cdots \leq t_n$ the increments $s(t_n) - s(t_{n-1})$, $s(t_{n-1}) - s(t_{n-2}), \cdots,$ $s(t_2) - s(t_1)$ are independent random variables.

- For $t_2 > t_1$, $s(t_2) - s(t_1) \sim \mathcal{N}(0, \sigma^2(t_2 - t_1))$ is equivalent to

$$s(t_2)\,|\,s(t_1) \sim \mathcal{N}(s(t_1), \sigma^2(t_2 - t_1))$$

  $s(t_2)$ given $s(t_1)$ is normally distributed with mean $s(t_1)$ and variance $\sigma^2(t_2 - t_1)$.

Brownian Motion III

Exercises:

1. Show that a standard brownian motion is intrinsically stationary
2. Show that a standard brownian motion is not weakly stationary

Ornstein-Uhlenbeck Process I

Definition (Ornstein-Uhlenbeck)

Let $s(t)$ be standard Brownian motion. The process

$$V(t) = e^{-t} s(e^{2t})$$

is called the Ornstein-Uhlenbeck process.

Ornstein-Uhlenbeck Process II

Show that $V(t)$ is intrinsically stationary and weakly stationary.

Brownian Motion and Ornstein-Uhlenbeck processes

Historical remarks:

- Robert Brown observes ceaseless irregular motion of small particles in a fluid. Motion explained by believing particles to be alive (1827-1829).
- Goul puts forward a kinetic theory to explain the motion; it is due to rapid bombardment of a huge number of fluid molecules (1860).
- Einstein presents the theory of “Brownian motion” (c. 1900). At the same time (c. 1900), Bachelier defines it to model stock options.
- The Ornstein-Uhlenbeck process was proposed by Uhlenbeck and Ornstein (1930) in a physical modelling context, as an alternative to Brownian Motion. The model has been used since in a wide variety of applications areas e.g. in finance (see Vasicek (1977) interest rate model).