Regression with B-splines

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Splines are polynomial segments joined end to end with segments contrained to be smooth at the joins. The points at which the segments join are called knots. Splines are defined by:

- order m = degree of polynomial + 1
- the locations of knots

For instance, the basis of splines of order $m = 1$ is piecewise constant and discontinuous such that:

$$\phi_{k,0}(t) = \begin{cases} 1 & t \in [u_k, u_{k+1}) \\ 0 & \text{otherwise} \end{cases}$$

with knots $\{u_k\}_{k=1,\cdots}$ such that $u_k < u_{k+1}$. Cox-de Boor recursion formula allows to compute the spline basis of function of degree p:

$$\phi_{k,p}(t) = \frac{t - u_k}{u_{k+p} - u_k}\,\phi_{k,p-1}(t) + \frac{u_{k+p+1} - t}{u_{k+p+1} - u_{k+1}}\,\phi_{k+1,p-1}(t) \tag{1}$$

1. Consider the four knots $u_0 = 0$, $u_1 = 1$, $u_2 = 2$, and $u_3 = 3$ defining the intervals (knot spans) $[0,1)$, $[1,2)$ and $[2,3)$ and the basis functions of degree 0 (order $m = 1$) $\phi_{0,0}(t) = 1$ on $[0,1)$ and 0 elsewhere, $\phi_{1,0}(t) = 1$ on $[1,2)$ and 0 elsewhere, and $\phi_{2,0}(t) = 1$ on $[2,3)$ and 0 elsewhere.
   (a) Draw basis funtions $\{\phi_{k,0}\}_{k=1,\cdots 2}$ on $[u_0, u_3]$.
   (b) Compute the spline basis ($\phi_{0,1}$ and $\phi_{1,1}$) of order $m = 2$ (with $p = 1$) using formula (1).
   (c) Draw basis funtions $\phi_{0,1}$ and $\phi_{1,1}$ on $[u_0, u_3]$.
   (d) Compute $\phi_{0,2}$ order $m = 3$ (with $p = 2$) using formula (1).
   (e) Draw $\phi_{0,2}$ on $[u_0, u_3]$.
2. Using formula (1), find a recursive formula for computing $\frac{D\,\phi_{k,p}(t)}{Dt}$ and $\frac{D^2\phi_{k,p}(t)}{D^2 t}$.
3. Compute $\frac{D^2\phi_{0,2}(t)}{D^2 t}$ and $\int_{u_0}^{u_3} \left(\frac{D^2\phi_{0,2}(t)}{D^2 t}\right)^2 dt$