State-Space Model I

In the Multivariate case:

- **State equation** with $\vec{s}_j$ a random vector

$$\vec{s}_j = \mathrm{F}\ \vec{s}_{j-1} + \mathrm{G}\ \vec{\epsilon}_j$$

  F is a square matrix, G is a matrix, and $\vec{\epsilon}_j \sim \mathcal{N}(0, \Sigma)$ is a random vector.

- **Observation equation**

$$\vec{o}_j = \mathrm{H}\ \vec{s}_j + \ \vec{e}_j$$

  with H a matrix and $\vec{e}_j \sim \mathcal{N}(0, \Sigma_o)$.

State-Space Model II

Given $p(\vec{s}_{j-1}|\vec{o}_{1:j-1}) = \mathcal{N}(\vec{s}_{j-1}; \mu_{j-1}, \Sigma_{j-1})$

- prediction

$$p(\vec{s}_j|\vec{o}_{1:j-1}) = \mathcal{N}(\vec{s}_j; \mathrm{F}\mu_{j-1}, \mathrm{R} = \mathrm{F}\Sigma_{j-1}\mathrm{F}^T + \mathrm{G}\Sigma\mathrm{G}^T)$$

- update $p(\vec{s}_j|\vec{o}_{1:j}) = \mathcal{N}(\vec{s}_j; \mu_j, \Sigma_j)$ with

$$\Sigma_j = \left(\mathrm{H}^T\Sigma_o^{-1}\mathrm{H} + \mathrm{R}^{-1}\right)^{-1}$$

  and

$$\mu_j = \Sigma_j\left(\mathrm{H}^T\Sigma_o^{-1}\vec{o}_j + \mathrm{R}^{-1}\mathrm{F}\mu_{j-1}\right)$$

State-Space Model III

Example

Consider the following model

1. Observation equation:

$$o_j = s_j + e_j$$

2. State equation

$$AR(3): \quad s_j = \phi_1\ s_{j-1} + \phi_2\ s_{j-2} + \phi_3\ s_{j-3} + \epsilon_j$$

   Define $\vec{s}_j$, $\vec{o}_j$, H, F and G .

State-Space Model IV

Answer: Using Matrix notation, the State equation is:

$$\underbrace{\begin{pmatrix} s_j \\ s_{j-1} \\ s_{j-2} \end{pmatrix}}_{\vec{s}_j} = \underbrace{\begin{bmatrix} \phi_1 & \phi_2 & \phi_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathrm{F}} \underbrace{\begin{pmatrix} s_{j-1} \\ s_{j-2} \\ s_{j-3} \end{pmatrix}}_{\vec{s}_{j-1}} + \underbrace{\begin{pmatrix} \epsilon_j \\ 0 \\ 0 \end{pmatrix}}_{\vec{\epsilon}_j}$$

and the observation equation is:

$$o_j = \underbrace{(1,\ 0,\ 0)}_{\mathrm{H}} \underbrace{\begin{pmatrix} s_j \\ s_{j-1} \\ s_{j-2} \end{pmatrix}}_{\vec{s}_j} + e_j$$

State-Space Model V

Exercises: Find F and G when s_j follows

1. an MA(2)
2. an ARMA(2,2)

State-Space Model VI

- F, G and H are assumed to be known and constant over time.
- More general state space models allow these matrices to depend on time.