$\{V_p\}$ in $\mathbb{C}^J$ (Slide #132, ST3454)

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We want to show that

$$V_1 = \begin{pmatrix} \exp\left(\imath\frac{2\pi\times 1\times 1}{J}\right) \\ \exp\left(\frac{\imath 2\pi\times 1\times 2}{J}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times 1\times J}{J}\right) \end{pmatrix} \quad V_2 = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times 2\times 1}{J}\right) \\ \exp\left(\frac{\imath 2\pi\times 2\times 2}{J}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times 2\times J}{J}\right) \end{pmatrix} \quad \cdots \quad V_J = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times J\times 1}{J}\right) \\ \exp\left(\frac{\imath 2\pi\times J\times 2}{J}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times J\times J}{J}\right) \end{pmatrix}$$

form an orthogonal basis of $\mathbb{C}^J$. We have J vectors $\{V_k\}_{k=\cdots J}$, we just need to show that they are independent from each other to be a basis function of the $J-$dimensional space $\mathbb{C}^J$.

Lets compute the scalar product between two vectors e_{k_1} , e_{k_2}:

$$< V_{k_1}, V_{k_2} >= \sum_{n=1}^{n=J} \exp\left(\frac{\imath 2\pi \times k_1 \times n}{J}\right) \exp\left(-\frac{\imath 2\pi \times k_2 \times n}{J}\right)$$

$$< V_{k_1}, V_{k_2} >= \sum_{n=1}^{n=J} \exp\left(\frac{\imath 2\pi \times (k_1 - k_2) \times n}{J}\right)$$

if $k_1 = k_2$ then

$$< V_{k_1}, V_{k_2} >= \sum_{n=1}^{n=J} \exp\left(\frac{\imath 2\pi \times 0 \times n}{J}\right) = J$$

Now we need to show that if $k_1 \neq k_2$ then $< V_{k_1}, V_{k_2} >= 0$. We note $K = (k_1 - k_2)$ and we rewrite:

$$\begin{aligned} < V_{k_1}, V_{k_2} > &= \textstyle\sum_{n=1}^{n=J} \exp\left(\frac{\imath 2\pi\times K\times n}{J}\right) \\ &= \textstyle\sum_{n=1}^{n=J} \left(\exp\left(\frac{\imath 2\pi\times K}{J}\right)\right)^n \end{aligned}$$

We recognize the form of a geometric series, so we can write (for $K \neq 0$):

$$\begin{aligned} < V_{k_1}, V_{k_2} > &= \exp\left(\frac{\imath 2\pi\times K}{J}\right) \left(\frac{1-\exp\left(\frac{\imath 2\pi\times K\times J}{J}\right)}{1-\exp\left(\frac{\imath 2\pi\times K}{J}\right)}\right) \\ &= \exp\left(\frac{\imath 2\pi\times K}{J}\right) \left(\frac{1-1}{1-\exp\left(\frac{\imath 2\pi\times K}{J}\right)}\right) \\ &= 0 \end{aligned}$$