Yuille pendulum: Ordinary Differential equations and time series

Prof. Rozenn Dahyot
Trinity College Dublin, Ireland

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Yule's pendulum. In 1927, in one of his article, Yule made a parallel between a dynamic system, a pendulum swinging, and autoregressive models. Concretely, the angular deviation (e.g. angle) $X(t)$ of the pendulum follows a dynamic equation, e.g. :

$$\ddot{X}(t) + 2\ \dot{X}(t) + 5\ X(t) = \epsilon(t) \tag{1}$$

where $\ddot{X}(t)$ is the second derivative of $X(t)$ (the acceleration), and $\dot{X}(t)$ is first order derivative (the velocity). $\epsilon(t)$ would represent unexpected perturbations or errors in the measurement process. t is the time.

1. We assume $\epsilon(t) = 0, \quad \forall t$. Verify that the function

   $$X(t) = A\ \exp(-t)\ \cos(2t), \quad \text{with } A \text{ is a constant} \tag{2}$$

   is a solution of the dynamic equation (1).

2. Equation (1) and its solution (2) are written with continuous notation. Now assume that we only observe $X(t)$ at regular interval h such that $X_n = X(n \times h)$ for $n = 0, \cdots, N$. In this case we have the following approximations:

   $$\ddot{X}(nh) \simeq \frac{X_n - 2X_{n-1} + X_{n-2}}{h^2}$$

   and

   $$\dot{X}(nh) \simeq \frac{X_n - X_{n-1}}{h}$$

   Using these approximations, rewrite equation (1) as an AutoRegressive model of order 2.

3. Excel Simulation. Create the following columns (cf. 1):

   - a column *time*, starting at 0 and incrementing 60 times by $+h$.
   - a column *Continuous y* that computes equation (2).
   - a column *AR(2) y* that computes the expression found in question 2. Initialise the Autoregressive model using the first two values computed with *Continuous y*.

   h is a parameter that can be chosen $h = 0.1$, and the constant A is chosen $A = 20$. Draw *Continuous y* and *AR(2) y* w.r.t. *time*. Comment on the graphs. What is the effect of increasing/decreasing the value of h? (hint: look at the formal definition of a derivative).

Time	Continuous y	AR(2) y
0	20	20
0.1	17.7	17.7
0.2		
⋮		

Table 1: Simulation in Excel: $A = 20$ and $h = .1$