Kriging techniques applied to times series

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1. Assume an Moving Average model of order 1, noted MA(1):

\[ s_{t+1} = \alpha \epsilon_t + \epsilon_{t+1} \quad \forall t \]  

(1)

where \(|\alpha| < 1\), \(\epsilon_t \sim \mathcal{N}(0, \sigma^2)\), \(\forall t\) (and i.i.d.) and notations \(s_{t+1} = s(t+\Delta)\) with \(s_t = s(t)\) where \(\Delta\) is the regular interval of time on the timeline between the random variables \(\{s_1, \cdots, s_J, s_{J+1}\}\) for which we have collected the first \(J\) observations \(\{s_1, \cdots, s_J\}\). Propose a Kriging solution for computing an estimate \(\hat{s}_{J+1}\) for r.v. \(s_{J+1}\), as well as its Kring variance associated with its uncertainty. Consider the following steps:

(a) Show that the stochastic process is stationary in mean i.e. \(\mathbb{E}[s_t] = \mu_s, \forall t\) (the mean remains constant overtime) and compute \(\mu_s\).

\textbf{Ans.} \(\mu_s = 0\)

(b) Show that the stochastic process is stationary in variance i.e. \(\mathbb{E}[(s_t - \mathbb{E}[s_t])^2] = \sigma^2_s, \forall t\) (the variance remains constant overtime) and compute \(\sigma^2_s\).

\textbf{Ans.} \(\sigma^2_s = (1 + \alpha^2)\sigma^2\)

(c) Compute the Covariance between \(s_t\) and \(s_{t-k}\), \(\forall k \in \mathbb{N}\).

\textbf{Ans.}

\[ \mathbb{C}[s_t, s_{t-k}] = \begin{cases} (1 + \alpha^2)\sigma^2, & k = 0 \\ \alpha\sigma^2, & k = 1 \\ 0, & \text{otherwise} \end{cases} \]

(d) \textbf{Ans.} The Kriging method suited is simple Kriging because the mean \(\mu_s\) is constant and known (this is 0). The solution lambdas can be computed (see slide 78 of lecture notes):

\[ \lambda = \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \vdots \\ \hat{\lambda}_J \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,J} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{J,1} & c_{J,2} & c_{J,3} & \cdots & c_{J,J} \end{pmatrix}^{-1} \begin{pmatrix} c_{1,J+1} \\ c_{2,J+1} \\ \vdots \\ c_{J,J+1} \end{pmatrix} \]

or

\[ \lambda = \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \vdots \\ \hat{\lambda}_J \end{pmatrix} = \begin{pmatrix} \sigma^2(1 + \alpha^2) & \alpha\sigma^2 & 0 & \cdots & 0 \\ \alpha\sigma^2 & \sigma^2(1 + \alpha^2) & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \sigma^2(1 + \alpha^2) & \alpha\sigma^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \alpha\sigma^2 \end{pmatrix} \]

\[ \mathbb{C}^{-1} \begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \vdots \\ \hat{\lambda}_J \end{pmatrix} = \begin{pmatrix} \alpha\sigma^2 \end{pmatrix} \]

\[ \hat{\lambda} = \begin{pmatrix} \alpha\sigma^2 \end{pmatrix} \]

(e) \textbf{Ans.} The kriging variance \(\mathbb{E}[\epsilon^2_{J+1}]\) can also be computed using the estimated \(\lambda s\) (e.g. see slide 73):

\[ \mathbb{E}[\epsilon^2_{J+1}] = \sigma^2(1 + \alpha^2) - 2\hat{\lambda}^T \hat{b} + \hat{\lambda}^T \mathbb{C}^{-1} \hat{\lambda} \]
2. Same exercise with an AR(1) model

\[ s_{t+1} = \alpha s_t + \epsilon_{t+1} \quad \forall \ t \tag{2} \]

where \(|\alpha| < 1, \epsilon_t \sim \mathcal{N}(0, \sigma^2), \forall \ t \) (and i.i.d.)

3. Same exercise with MA(1) with an intercept:

\[ s_{t+1} = \alpha_0 + \alpha \epsilon_t + \epsilon_{t+1} \quad \forall \ t \tag{3} \]

where \(|\alpha| < 1, \epsilon_t \sim \mathcal{N}(0, \sigma^2), \forall \ t \) (and i.i.d.)

4. Same exercise but assume an MA(2) stationary in mean and variance (this time you are not given constraint on parameters e.g. \(|\alpha| < 1 \) but instead you know it is stationary in mean & variance).

5. Same exercise but assume an AR(2) stationary in mean and variance (this time you are not given constraint on parameters e.g. \(|\alpha| < 1 \) but instead you know it is stationary in mean & variance).

6. Same exercise but assume an ARMA(1,1) stationary in mean and variance (this time you are not given constraint on parameters e.g. \(|\alpha| < 1 \) but instead you know it is stationary in mean & variance).