Kriging techniques applied to times series

Prof. Rozenn Dahyot
Trinity College Dublin, Ireland

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1. Assume an Moving Average model of order 1, noted MA(1):

$$s_{t+1} = \alpha\, \epsilon_t + \epsilon_{t+1} \quad \forall t \tag{1}$$

where $|\alpha| < 1$, $\epsilon_t \sim \mathcal{N}(0, \sigma^2), \forall t$ (and i.i.d.) and notations $s_{t+1} = s(t + \Delta)$ with $s_t = s(t)$ where Δ is the regular interval of time on the timeline between the random variables $\{s_1, \cdots, s_J, s_{J+1}\}$ for which we have collected the first J observations $\{s_1^{(1)}, \cdots, s_J^{(1)}\}$. Propose a Kriging solution for computing an estimate $\hat{s}_{J+1}$ for r.v. s_{J+1}, as well as its Krigin variance associated with its uncertainty. Consider the following steps:

(a) Show that the stochastic process is stationary in mean i.e. $\mathbb{E}[s_t] = \mu_s, \forall t$ (the mean remains constant overtime) and compute μ_s.

Ans. $\mu_s = 0$

(b) Show that the stochastic process is stationary in variance i.e. $\mathbb{E}\left[(s_t - \mathbb{E}[s_t])^2\right] = \sigma_s^2, \forall t$ (the variance remains constant overtime) and compute σ_s^2.

Ans. $\sigma_s^2 = (1 + \alpha^2)\sigma^2$

(c) Compute the Covariance between s_t and s_{t-k}, $\forall k \in \mathbb{N}$.

Ans.

$$\mathbb{C}[s_t, s_{t-k}] = \begin{cases} (1+\alpha^2)\sigma^2, k = 0 \\ \alpha\sigma^2, k = 1 \\ 0 \quad \text{otherwise} \end{cases}$$

(d) **Ans.** The Kriging method suited is simple Kriging because the mean μ_s is constant and known (this is 0). The solution lambdas can be computed (see slide 78 of lecturenotes):

$$\begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \vdots \\ \hat{\lambda}_J \end{pmatrix} = \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,J} \\ c_{2,1} & c_{2,2} & c_{2,3} & \cdots & c_{2,J} \\ \vdots & & & & \\ c_{J,1} & c_{J,2} & c_{J,3} & \cdots & c_{J,J} \end{bmatrix}^{-1} \begin{pmatrix} c_{1,J+1} \\ c_{2,J+1} \\ \vdots \\ c_{J,J+1} \end{pmatrix}$$

or

$$\underbrace{\begin{pmatrix} \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \vdots \\ \hat{\lambda}_J \end{pmatrix}}_{\vec{\lambda}} = \underbrace{\begin{bmatrix} \sigma^2(1+\alpha^2) & \alpha\sigma^2 & 0 & \cdots & 0 \\ \alpha\sigma^2 & \sigma^2(1+\alpha^2) & \alpha\sigma^2 & 0 & 0 \\ \vdots & \ddots & & & \\ 0 & 0 & 0 & \cdots & \sigma^2(1+\alpha^2) \end{bmatrix}^{-1}}_{\mathrm{C}^{(-1)}} \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \alpha\sigma^2 \end{pmatrix}}_{\vec{b}}$$

(e) **Ans.**The kriging variance $\mathbb{E}[\epsilon_{J+1}^2]$ can also be computed using the estimated λs (e.g. see slide 73):

$$\mathbb{E}[\epsilon_{J+1}^2] = \sigma^2(1+\alpha^2) - 2\vec{\hat{\lambda}}^T\vec{b} + \vec{\hat{\lambda}}^T \mathrm{C}\vec{\hat{\lambda}}$$

2. Same exercise with an AR(1) model
$$s_{t+1} = \alpha\ s_t + \epsilon_{t+1} \quad \forall t \tag{2}$$
where $|\alpha| < 1$, $\epsilon_t \sim \mathcal{N}(0,\sigma^2), \forall t$ (and i.i.d.)

3. Same exercise with MA(1) with an intercept:
$$s_{t+1} = \alpha_0 + \alpha\ \epsilon_t + \epsilon_{t+1} \quad \forall t \tag{3}$$
where $|\alpha| < 1$, $\epsilon_t \sim \mathcal{N}(0,\sigma^2), \forall t$ (and i.i.d.

4. Same exercise but assume an MA(2) stationary in mean and variance (this time you are not given constraint on parameters e.g. $|\alpha| < 1$ but instead you know it is stationary in mean & variance).

5. Same exercise but assume an AR(2) stationary in mean and variance (this time you are not given constraint on parameters e.g. $|\alpha| < 1$ but instead you know it is stationary in mean & variance).

6. Same exercise but assume an ARMA(1,1) stationary in mean and variance (this time you are not given constraint on parameters e.g. $|\alpha| < 1$ but instead you know it is stationary in mean & variance).