Transforms I

Definition (Integral transform)

An integral transform $\mathcal{T}$ is defined as:

$$\mathcal{T}[f(t)] \equiv F(u) = \int_{t_1}^{t_2} K(t,u)\ f(t)\ dt$$

where K is the kernel function. This can also be understood as a linear combination ('continuous sum') over a basis of functions K.

Example

- The Laplace transform is an example of integral transform with $K(t,u) = \exp(-u\ t)$.
- The Fourier transform is another example with $K(t,u) = \exp(-\jmath u\ t)$.

Transforms II

Another useful example is the Radon transform:

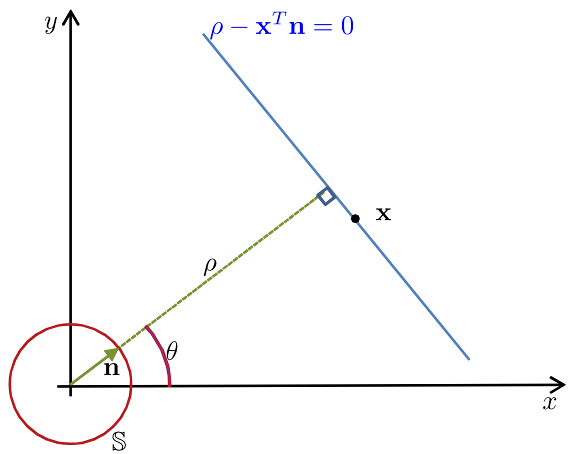
Definition (Radon transform)

Having a function f defined on a domain $\mathbf{x} = (x,y) \in \mathbb{R}^2$, the Radon transform of f is its integral along the line of equation $\rho - x\cos\theta - y\sin\theta = 0$ i.e.:

$$\begin{aligned} Rf(\rho,\theta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \delta(\rho - x\cos\theta - y\sin\theta)\ f(x,y)\ dx\ dy \\ &= \int_{\mathbb{R}^2}\ \delta(\rho - \mathbf{x}^T\mathbf{n})\ f(\mathbf{x})\ d\mathbf{x} \end{aligned}$$

with $\delta(\cdot)$ the Dirac delta function, $\rho \in \mathbb{R}$, $\theta \in [-\frac{\pi}{2};\frac{\pi}{2}]$ and $\mathbf{n} = (\cos\theta, \sin\theta)^T$.

Transforms III



Transforms IV

Exercises:

1. Assuming that $f(\mathbf{x}) = p_{\mathbf{x}}(\mathbf{x})$ is the probability density function of $\mathbf{x}$, what is $Rf(\rho,\theta)$ when $\theta = 0$ and $\theta = \frac{\pi}{2}$?
2. Assuming that $f(\mathbf{x}) = p_{\mathbf{x}}(\mathbf{x})$ is the probability density function of $\mathbf{x}$, is $Rf(\rho,\theta)$ probability density function?

Application Radon Transform I

1. Consider the r.v. $\mathbf{x} = (x, y)$ for which only one observation $\mathbf{x}^{(1)}$ is available. Then the empirical density function is:

$$\hat{p}_{\mathbf{x}}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}^{(1)}) = \delta(x - x^{(1)})\ \delta(y - y^{(1)})$$

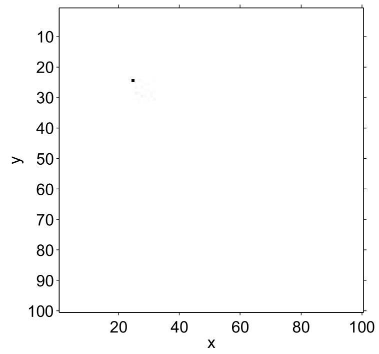
   What is the Radon transform of $p_{\mathbf{x}}(\mathbf{x})$?

2. With two observations

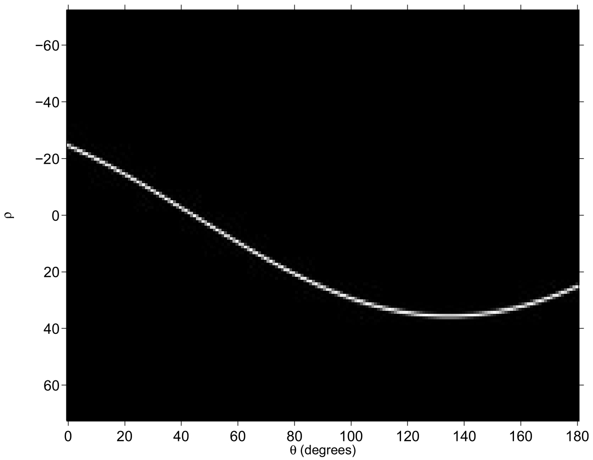
$$\hat{p}_{\mathbf{x}}(\mathbf{x}) = \frac{1}{2}\left[\delta(\mathbf{x} - \mathbf{x}^{(1)}) + \delta(\mathbf{x} - \mathbf{x}^{(2)})\right]$$

   What is the Radon transform of $p_{\mathbf{x}}(\mathbf{x})$?

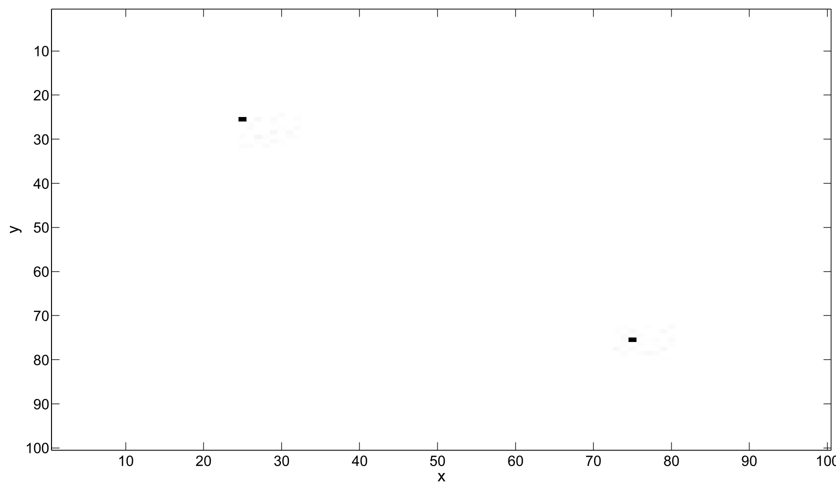
Application Radon Transform II



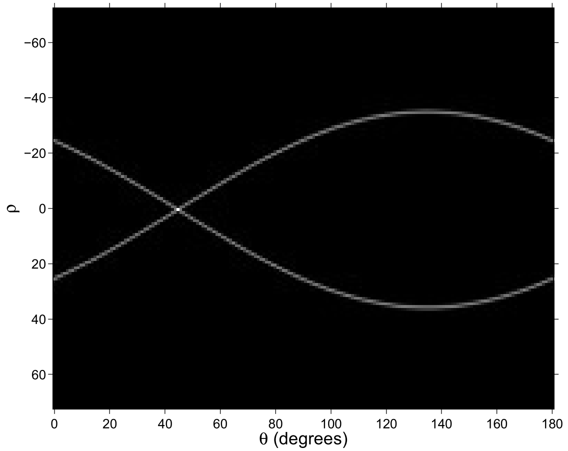
Application Radon Transform III



Application Radon Transform IV



Application Radon Transform V



Conclude on how the Radon transform can be used to detect perfect lineament.

Application Radon Transform VI

Remarks:

- The *Hough transform* is a very closely related method to the Radon transform to find shapes (line, circles, etc.).
- The formulation of the Radon tranform itself has been generalised to consider quadratic shapes for instance:
  The general quadratic Radon transform, Koen Denecker, Jeroen Van Overloop and Frank Sommen, Inverse Problems 14 (1998) 615-633

Application Radon Transform VII

Example: Archeology

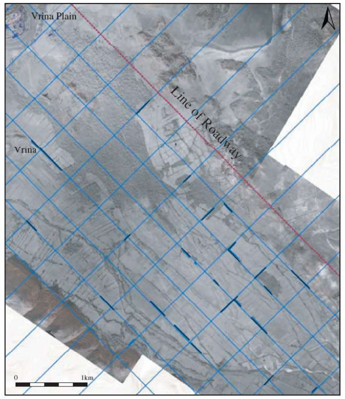
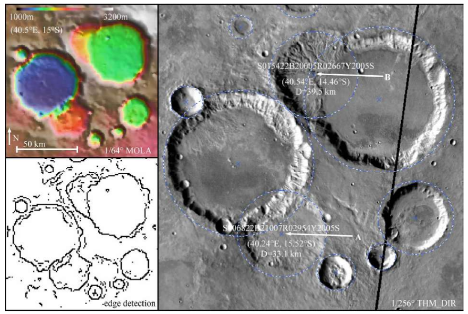


Fig. 5. Alignments detected within the study area (dark lines) conforming to a possible 12 by 16 *actus* grid layout (light lines).

Detecting Roman land boundaries in aerial photographs using Radon transforms, D.J. Bescoby, Journal of Archaeological Science Volume 33, Issue 5, May 2006, Pages 735-743

Application Radon Transform VIII

Example: crater detection on Mars -



Method for Crater Detection From Martian Digital Topography Data Using Gradient Value/Orientation, Morphometry, Vote Analysis, Slip Tuning, and Calibration, Goran Salamuniccar and Sven Loncaric, IEEE transactions on Geoscience and remote sensing, vol 48, n. 5, May 2010