**Transforms I**

**Definition (Integral transform)**

An integral transform $T$ is defined as:

$$ T[f(t)] = F(u) = \int_{t_1}^{t_2} K(t, u) f(t) \, dt $$

where $K$ is the kernel function. This can also be understood as a linear combination ('continuous sum') over a basis of functions $K$.

**Example**

- The Laplace transform is an example of integral transform with $K(t, u) = \exp(-u \cdot t)$.
- The Fourier transform is another example with $K(t, u) = \exp(-ju \cdot t)$.

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**Transforms II**

**Definition (Radon transform)**

Another useful example is the Radon transform:

Having a function $f$ defined on a domain $x = (x, y) \in \mathbb{R}^2$, the Radon transform of $f$ is its integral along the line of equation $\rho - x \cos \theta - y \sin \theta = 0$ i.e.:

$$ Rf(\rho, \theta) = \int_{\mathbb{R}} \int_{\mathbb{R}} \delta(\rho - x \cos \theta - y \sin \theta) \, f(x, y) \, dx \, dy $$

$$ = \int_{\mathbb{R}^2} \delta(\rho - x^T \mathbf{n}) \, f(x) \, dx $$

with $\delta(\cdot)$ the Dirac delta function, $\rho \in \mathbb{R}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\mathbf{n} = (\cos \theta, \sin \theta)^T$.

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**Transforms III**

**Transforms IV**

**Exercises:**

- Assuming that $f(x) = p_x(x)$ is the probability density function of $x$, what is $Rf(\rho, \theta)$ when $\theta = 0$ and $\theta = \frac{\pi}{2}$?

- Assuming that $f(x) = p_x(x)$ is the probability density function of $x$, is $Rf(\rho, \theta)$ probability density function?
Application Radon Transform I

1. Consider the r.v. $x = (x, y)$ for which only one observation $x^{(1)}$ is available. Then the empirical density function is:

$$\hat{p}_x(x) = \delta(x - x^{(1)}) = \delta(x - x^{(1)}) \delta(y - y^{(1)})$$

What is the Radon transform of $p_x(x)$?

2. With two observations

$$\hat{p}_x(x) = \frac{1}{2} \left[ \delta(x - x^{(1)}) + \delta(x - x^{(2)}) \right]$$

What is the Radon transform of $p_x(x)$?
Conclude on how the Radon transform can be used to detect perfect lineament.

Remarks:

- The Hough transform is a very closely related method to the Radon transform to find shapes (line, circles, etc.).

- The formulation of the Radon transform itself has been generalised to consider quadratic shapes for instance:

Example: Archeology


Example: crater detection on Mars