Kalman Filter I

- We have illustrated the idea behind Filtering with a transition model of order 1 (This can be extended to any order). In practice, we need to be able to calculate or compute \( p(s_j|a_{1:j}) \). Kalman filtering is one elegant way of doing that.
- In 1960, R.E. Kalman published his famous paper describing a recursive solution to the discrete-data linear filtering problem.
- The Kalman filter uses hypotheses of linearity (in the state transition equation, and the observation equation) and Normal distributions.

Kalman Filter II

Exercise: Given

\[
\begin{align*}
\ p(s_1) &= \mathcal{N}(s_1; \mu_1, \sigma_1^2) \quad \text{(initial state pdf)} \\
\ p(s_j|s_{j-1}) &= \mathcal{N}(s_j; \alpha s_{j-1}, \sigma_j^2) \quad \text{(state transition pdf)} \\
\ p(o_j|s_j) &= \mathcal{N}(o_j; \kappa_j, \sigma_o^2) \quad \text{(observation pdf)}
\end{align*}
\]

with the notation \( \mathcal{N}(x; \mu, \sigma^2) \) corresponding to the normal pdf for r.v. \( x \) with mean \( \mu \) and variance \( \sigma^2 \). Show by induction that \( p(s_j|a_{1:j}) \) is a normal distribution \( \mathcal{N}(s_j; \mu_j, \sigma_j^2) \) and give the recursive formula to compute parameters \((\mu_j, \sigma_j^2)\) assuming \( p(s_{j-1}|a_{1:j-1}) = \mathcal{N}(s_{j-1}; \mu_{j-1}, \sigma_{j-1}^2) \).

Kalman Filter III

Answer.

1. at \( j = 1 \), we know the initial state \( s_1 \) is normally distributed (given in the hypothesis).
2. assuming at \( j - 1 \) that \( p(s_{j-1}|a_{1:j-1}) = \mathcal{N}(s_{j-1}; \mu_{j-1}, \sigma_{j-1}^2) \), then solving the prediction and update steps of filtering, we find that:

\[
\begin{align*}
\mu_j &= \alpha (1 - K_j) \mu_{j-1} + K_j o_j \\
\sigma_j^2 &= \sigma_o^2 K_j \\
K_j &= \frac{\sigma_o^2 \sigma_j^2}{\sigma_o^2 + \alpha^2 \sigma_{j-1}^2}
\end{align*}
\]

Kalman Filter IV

1. the prediction step corresponds to (using result \( \langle N_A|N_B \rangle \)):

\[
\begin{align*}
\ p(s_j|a_{1:j-1}) &= \int \mathcal{N}(s_j; \alpha s_{j-1}, \sigma_j^2) \mathcal{N}(s_{j-1}; \mu_{j-1}, \sigma_{j-1}^2) \ ds_{j-1} \\
&= \frac{1}{\sigma_{j-1}} \int \mathcal{N} \left( s_j; \frac{\alpha \mu_{j-1}}{\alpha^2 + \sigma_{j-1}^2}, \frac{\sigma_j^2 + \sigma_{j-1}^2}{\sigma_{j-1}} \right) \ ds_{j-1} \\
&= \frac{1}{\sigma_{j-1}} \mathcal{N} \left( 0; \frac{\alpha \mu_{j-1}}{\alpha^2 + \sigma_{j-1}^2}, \frac{\sigma_j^2 + \sigma_{j-1}^2}{\sigma_{j-1}} \right) \\
&= \mathcal{N}(s_j; \alpha \mu_{j-1}, \sigma^2 + \alpha^2 \sigma_{j-1}^2)
\end{align*}
\]

2. update (rewriting is required to be able to identify \((\mu_j, \sigma_j^2)\)):

\[
\begin{align*}
\ p(s_j|a_{1:j}) &= \mathcal{N}(o_j; s_j, \sigma_o^2) \mathcal{N}(s_j; \alpha \mu_{j-1}, \sigma^2 + \alpha^2 \sigma_{j-1}^2) \\
&= \mathcal{N}(s_j; \mu_j, \sigma_j^2)
\end{align*}
\]
Kalman Filter V

- These two parameters \((\mu_j, \sigma_j^2)\) are enough to characterise fully the Normal distribution \(p(s_j| \alpha_{1:j})\).

- The observations available \(\{s_j^{(1)}\}\) are plugged in the expression \((\mu_j, \sigma_j^2)\) so that they can be computed iteratively.