Fourier transform I

The Fourier transform is an important tool to analyse signals, discrete or continuous.

A Wavelet tour of signal processing , by S. Mallat, second edition

Definition (Fourier transform)

The Fourier integral measures *how much* oscillations at the frequency ω there is the function f:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\ \exp^{(-\imath\omega t)}\ dt$$

Fourier transform II

Definition (Inverse Fourier transform)

The Inverse Fourier transform can be computed:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\ \exp^{(\imath\omega t)}\ d\omega$$

Exercises: Fourier transform

Compute the Fourier transforms of:

1. $$f(t) = \begin{cases} 1 & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$
2. $f(t) = \delta(t)$
3. $f(t) = \delta(t - t_0)$

Fourier transform III

4. $f(t) = \exp(-at)\ h(t)$ (h is the heaviside step function)
5. $f(t) = \cos(\omega_0 t)$
6. Show
    1. Linearity:
    $$a\, x(t) + b\, y(t) \leftrightarrow a\ X(\omega) + b\ Y(\omega)$$
    2. Time shifting:
    $$x(t - t_0) \leftrightarrow \exp(-\imath\omega t_0) X(\omega)$$

Fourier transform IV

Definition (Convolution)

The convolution of two functions $x(t)$ and $h(t)$ gives a function $y(t)$ such that:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\ h(t - \tau)\ d\tau$$

Theorem (Convolution)

If $y(t) = x(t) * h(t)$ *then:*

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

Prove this theorem.

Two-dimensional Fourier transform I

Definition (2D Fourier Transform)

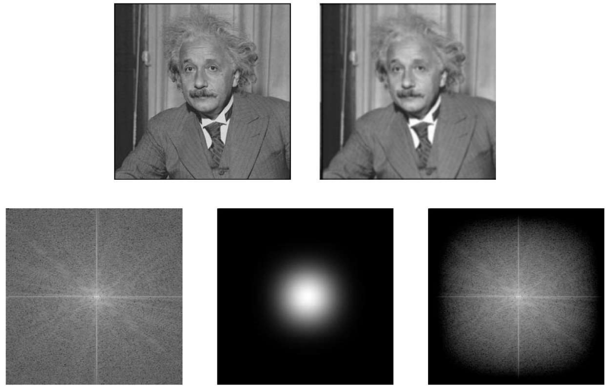
$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ \exp^{(-\imath \omega_x x)}\ \exp^{(-\imath \omega_y y)}\ dx\ dy$$

Remember that Spectral analysis is well suited for signal that is observed at regular interval. In particular, 2D Fourier Transform is suitable to analyse images:

Definition (image)

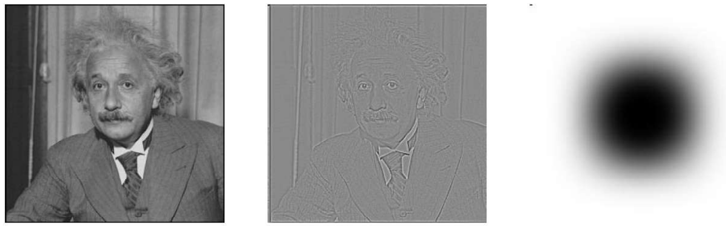
An image can be understood as a stochastic process $s(x)$ that has been observed at J locations $\{x_j\}_{j=1,\cdots,J}$ lying on a regular 2D grid. For grey level images, the state space of s is all integers between 0 (black) and 255 (white).

Two-dimensional Fourier transform II



Low pass filtering

Two-dimensional Fourier transform III



High pass filtering