Answers to Exercises on ODE (31/10/2007)

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1. Solve $y' + 2y = \exp(-t)$.

solution:

This is an ordinary differential equation, first order and linear. It is not separable as I cannot put all the elements depending on t on one side of the sign of equality and the elements depending on y on the other side:

$$\begin{array}{lcl} y' + 2y = \exp(-t) & \equiv & \frac{dy}{dt} + 2y = \exp(-t) \\ & \equiv & dy + 2y\, dt = \exp(-t)\, dt \end{array}$$

The equation can be written $y' + p(t)y = q(t)$ with $p(t) = 2$ and $q(t) = \exp(-t)$. We compute the solution y by noticing that y is equivalently solution of:

$$\frac{d\left(y \exp\left(\int_a^t p(s)ds\right)\right)}{dt} = q(t) \exp\left(\int_a^t p(s)ds\right)$$

or

$$\frac{d\left(y \exp\left(\int_a^t 2ds\right)\right)}{dt} = \exp(-t) \exp\left(\int_a^t 2ds\right)$$

or

$$\frac{d\left(y \exp(2t)\ \exp(-2a)\right)}{dt} = \exp(-t) \exp(2t)\ \exp(-2a)$$

simplified to (the constant $\exp(-2a)$ is removed on both side):

$$\frac{d\left(y \exp(2t)\ \right)}{dt} = \exp(-t) \exp(2t)$$

we can now integrate on both side in between a and τ:

$$\int_a^\tau \frac{d\left(y \exp(2t)\ \right)}{dt} dt = \int_a^\tau \exp(-t) \exp(2t) dt$$

or

$$\int_a^\tau d\left(y \exp(2t)\ \right) = \int_a^\tau \exp(-t) \exp(2t) dt$$

or

$$y(\tau) \exp(2\tau) - y(a) \exp(2a) = \exp(\tau) - \exp(2a)$$

or

$$y(\tau) = \exp(-\tau) + (-\exp(2a) + y(a) \exp(2a)) \exp(-2\tau)$$

or (changing the name of the variable $\tau = t$):

$$\begin{array}{ll} y(t) & = \exp(-t) + (-\exp(2a) + y(a) \exp(2a)) \exp(-2t) \\ & = \exp(-t) + B \exp(-2t) \end{array}$$

Now you can verify that this function verifies the original ODE.

2. Solve $y' = -3y$.

solution:

This is an ordinary differential equation, <u>first order</u> and <u>linear</u>. It is <u>separable</u>:

$$y' = -3y \qquad \equiv \qquad \tfrac{dy}{dt} = -3y$$

$$\equiv \qquad \tfrac{dy}{y} = -3\ dt$$

We can integrate both side in between $t = a$ and $t = \tau$ and we get:

$$\log(y(\tau)) - \ln(y(a)) = -3(\tau - a)$$

taking the exponential it gives:

$$\frac{y(\tau)}{y(a)} = \exp(-3(\tau - a))$$

or

$$y(\tau) = y(a) \cdot \exp(-3(\tau - a))$$

$y(a)$ is a constant and represent the initial condition.

3. Solve $y' - ty = t^2y^2$.

solution:

This is an ordinary differential equation, <u>first order</u> and <u>not linear</u>. It is a <u>Bernouilli</u> equation. We change variable $w = \frac{1}{y}$ then the equation becomes:

$$y' - ty = t^2y^2 \qquad \equiv \qquad w' + tw = -t^2$$

which is a linear 1st order equation that can be solved:

$$w(t) = \left(w(a) \exp\left(-\frac{a^2}{2}\right) + \int_a^t s^2 \exp\left(-\frac{s^2}{2}\right) ds \right) \cdot \exp\left(\frac{t^2}{2}\right)$$

Some integrals are difficult to solve (e.g. $\int_o^t s^2 \exp\left(-\frac{s^2}{2}\right) ds$). Now the last operation to perform is to compute $y(t) = 1/w(t)$.