Answers to Exercises on Fourier transform (5-6/12/2007)

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1 Computation of Fourier transform

1.
$$f(t) = \begin{cases} 1 & -T < t < T \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad F(\omega) = \int_{-T}^{+T} \exp(-\imath\omega t)\, dt$$
$$\leftrightarrow \quad F(\omega) = [\tfrac{\exp(-\imath\omega t)}{-\imath\omega}]_{-T}^{+T}$$
$$\leftrightarrow \quad F(\omega) = \tfrac{2\sin(\omega T)}{\omega}$$

2.
$$f(t) = \delta(t) \quad \leftrightarrow \quad F(\omega) = \int_{-\infty}^{+\infty} \delta(t) \exp(-\imath\omega t)\, dt$$
$$F(\omega) = \exp(-\imath\omega\; 0) = 1$$

3.
$$f(t) = \delta(t-t_0) \quad \leftrightarrow \quad F(\omega) = \int_{-\infty}^{+\infty} \delta(t-t_0) \exp(-\imath\omega t)\, dt$$
$$F(\omega) = \exp(-\imath\omega t_0)$$

4.
$$f(t) = h(t)\exp(-at) \quad \text{with } a>0 \quad \leftrightarrow \quad F(\omega) = \int_{-\infty}^{+\infty} h(t)\exp(-at)\exp(-\imath\omega t)\, dt$$
$$F(\omega) = \int_{0}^{+\infty} \exp(-(a+\imath\omega)t)\, dt$$
$$F(\omega) = \left[\tfrac{-\exp(-(a+\imath\omega)t)}{(a+\imath\omega)}\right]_0^{+\infty}$$
$$F(\omega) = \tfrac{1}{(a+\imath\omega)}$$

Remark: the limit of $\frac{-\exp(-(a+\imath\omega)t)}{(a+\imath\omega)}$ is 0 when t goes to ∞.

5. For $f(t) = \cos(\omega_0 t)$, we can notice that the inverse Fourier transform of $\delta(\omega-\omega_0)$ is $\frac{1}{2\pi}\exp(\imath\omega_0 t)$. We know:

$$f(t) = \cos(\omega_0 t) = \frac{\exp(\imath\omega_0 t) + \exp(-\imath\omega_0 t)}{2}$$

so consequently (using the property of linearity for the Fourier transform):

$$F(w) = \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$$

2 Prove the Time shifting property of the Fourier transform

We want to show that the Fourier transform of a signal $x(t-t_0)$ is $\exp(-\imath\omega t_0)X(\omega)$.

$$f(t) = x(t-t_0) \quad \leftrightarrow \quad F(\omega) = \int_{-\infty}^{+\infty} x(t-t_0)\; \exp(-\imath\omega t)\, dt$$

we change the variable $\tau = t - t_0$ and the integral becomes:

$$f(t) = x(t-t_0) \quad \leftrightarrow \quad F(\omega) = \int_{-\infty}^{+\infty} x(\tau)\; \exp(-\imath\omega(\tau+t_0))\, d\tau$$
$$\leftrightarrow \quad F(\omega) = \exp(-\imath\omega t_0) \int_{-\infty}^{+\infty} x(\tau)\; \exp(-\imath\omega\tau)\, d\tau$$
$$\leftrightarrow \quad F(\omega) = \exp(-\imath\omega t_0) X(\omega)$$

where $X(\omega)$ is the Fourier transform of $x(t)$.

3 Proof of the Convolution theorem

Lets consider 3 functions such that $y(t) = x(t) * h(t)$. The Fourier transform of $y(t)$ is computed by:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad \leftrightarrow \qquad Y(\omega) = \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right) \exp(-\imath\omega t)\, dt$$

Removing the brackets in the integral and using the commutativity of the multiplication, and also the time shifting property, we have a double integral to solve:

$$\begin{aligned}
y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \qquad &\leftrightarrow \qquad Y(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\, \exp(-\imath\omega t)\, d\tau\, dt \\
&\leftrightarrow \qquad Y(\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\, \exp(-\imath\omega t)\, dt\, d\tau \\
&\leftrightarrow \qquad Y(\omega) = \int_{-\infty}^{+\infty} x(\tau) \left(\int_{-\infty}^{+\infty} h(t-\tau)\, \exp(-\imath\omega t)\, dt\right)\, d\tau \\
&\leftrightarrow \qquad Y(\omega) = \int_{-\infty}^{+\infty} x(\tau) H(\omega) \exp(-i\omega\tau) d\tau \\
&\leftrightarrow \qquad Y(\omega) = H(\omega) \int_{-\infty}^{+\infty} x(\tau) \exp(-i\omega\tau) d\tau \\
&\leftrightarrow \qquad Y(\omega) = H(\omega)\, X(\omega)
\end{aligned}$$

4 Basis of $\mathbb{C}^N$

We want to show that

$$e_1 = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times 1\times 1}{N}\right) \\ \exp\left(\frac{\imath 2\pi\times 1\times 2}{N}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times 1\times N}{N}\right) \end{pmatrix} \quad e_2 = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times 2\times 1}{N}\right) \\ \exp\left(\frac{\imath 2\pi\times 2\times 2}{N}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times 2\times N}{N}\right) \end{pmatrix} \quad \cdots \quad e_N = \begin{pmatrix} \exp\left(\frac{\imath 2\pi\times N\times 1}{N}\right) \\ \exp\left(\frac{\imath 2\pi\times N\times 2}{N}\right) \\ \vdots \\ \exp\left(\frac{\imath 2\pi\times N\times N}{N}\right) \end{pmatrix}$$

form an orthogonal basis of $\mathbb{C}^N$. We have N vectors $\{e_k\}_{k=\cdots N}$, we just need to show that they are independent from each other to be a basis function of the $N-$dimensional space $\mathbb{C}^N$.

Lets compute the scalar product between two vectors e_{k_1} , e_{k_2}:

$$< e_{k_1}, e_{k_2} >= \sum_{n=1}^{n=N} \exp\left(\frac{\imath 2\pi \times k_1 \times n}{N}\right) \exp\left(-\frac{\imath 2\pi \times k_2 \times n}{N}\right)$$

Remark: the minus in the second exponential function comes from the definition of the scalar product on complex vectors.

$$< e_{k_1}, e_{k_2} >= \sum_{n=1}^{n=N} \exp\left(\frac{\imath 2\pi \times (k_1 - k_2) \times n}{N}\right)$$

if $k_1 = k_2$ then

$$< e_{k_1}, e_{k_2} >= \sum_{n=1}^{n=N} \exp\left(\frac{\imath 2\pi \times 0 \times n}{N}\right) = N$$

Now we need to show that if $k_1 \neq k_2$ then $< e_{k_1}, e_{k_2} >= 0$. We note $K = (k_1 - k_2)$ and we rewrite:

$$\begin{aligned}
< e_{k_1}, e_{k_2} > &= \sum_{n=1}^{n=N} \exp\left(\frac{\imath 2\pi\times K\times n}{N}\right) \\
&= \sum_{n=1}^{n=N} \left(\exp\left(\frac{\imath 2\pi\times K}{N}\right)\right)^n
\end{aligned}$$

We recognize the form of a geometric series, so we can write (for $K \neq 0$):

$$
\begin{aligned}
< e_{k_1}, e_{k_2} > \quad &= \exp\left(\frac{\imath 2\pi \times K}{N}\right)\left(\frac{1-\exp\left(\frac{\imath 2\pi \times K \times N}{N}\right)}{1-\exp\left(\frac{\imath 2\pi \times K}{N}\right)}\right) \\
&= \exp\left(\frac{\imath 2\pi \times K}{N}\right)\left(\frac{1-1}{1-\exp\left(\frac{\imath 2\pi \times K}{N}\right)}\right) \\
&= 0
\end{aligned}
$$