CS7ET01: Mathematics

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This document refers to questions asked in the lecturenotes of the 21 and 22 of November 2007.

1 Using the Least squares method to solve $y' + y = 0$

The first exercise is about finding the coefficients a_1 and a_2 of the trial solution $\tilde{y}(t) = 1 + a_1\, t + a_2\, t^2$ for the differential equation $y' + y = 0$, using the Least squares method.

The residual function is defined as:

$$R(t;\mathbf{a}) = 1 + (1+t)\, a_1 + (2t + t^2)\, a_2 \tag{1}$$

Using the least squares method to find a_1 and a_2 gives us the following system to solve:

$$(S) \equiv \begin{cases} \int_0^1 \frac{\partial R(t;\mathbf{a})}{\partial a_1}\, R(t;\mathbf{a})\, dt = 0 \\ \\ \int_0^1 \frac{\partial R(t;\mathbf{a})}{\partial a_2}\, R(t;\mathbf{a})\, dt = 0 \end{cases} \tag{2}$$

First let express the partial derivatives of $R(t;a)$ (using equation (1)):

$$\begin{cases} \frac{\partial R(t;\mathbf{a})}{\partial a_1} = 1 + t \\ \\ \frac{\partial R(t;\mathbf{a})}{\partial a_2} = 2t + t^2 \end{cases}$$

Replacing R and its derivatives in equations (2) gives the system:

$$(S) \equiv \begin{cases} \int_0^1 (1+t)\, (1 + (1+t)\, a_1 + (2t+t^2)\, a_2)\, dt = 0 \\ \\ \int_0^1 (2t+t^2)\, (1 + (1+t)\, a_1 + (2t+t^2)\, a_2)\, dt = 0 \end{cases}$$

which is equivalent to

$$(S) \equiv \begin{cases} \frac{3}{2} + \frac{7}{3} a_1 + \frac{9}{4} a_2 = 0 \\ \\ \frac{4}{3} + \frac{9}{4} a_1 + \frac{38}{15} a_2 = 0 \end{cases}$$

Solving that system gives $a_1 = -0.9427$ and $a_2 = 0.3110$ and the trial solution to the differential equation $y' + y = 0$, is:

$$\tilde{y}(t) = 1 - 0.9427\, t + 0.3110\, t^2$$

2 Solving $y'' + y = 1$ with $y(0) = 1$ and $y(1) = 0$ on the interval $[0; 1]$

Use the lecturenotes 07/11/2007.

2.1 Analytical solutions

2.1.1 Using Laplace transform (easiest)

The Laplace transform of the differential equation $y'' + y = 1$ is:

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{1}{s}$$

or

$$Y(s) = \frac{1}{s(s^2+1)} + \frac{s}{s^2+1}y(0) + \frac{1}{s^2+1}y'(0)$$

Using the table of conversion for computing the inverse of Laplace transform, we have:

$$\frac{s}{s^2+1} \quad \rightarrow \quad \cos(t)$$

and

$$\frac{1}{s^2+1} \quad \rightarrow \quad \sin(t)$$

For the term $\frac{1}{s(s^2+1)}$, we rewrite it first as:

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} \quad \rightarrow 1 - \cos(t)$$

So we can now conclude that:

$$y(t) = 1 - \cos(t) + y(0)\cos(t) + y'(0)\sin(t)$$

we know $y(0) = 1$ so

$$y(t) = 1 + y'(0)\sin(t)$$

we also know that $y(1) = 0$:

$$y(1) = 1 + y'(0)\sin(1)$$

therefore $y'(0) = \frac{-1}{\sin(1)}$ and

$$y(t) = 1 - \frac{\sin(t)}{\sin(1)}$$

Now you can verify that we do have $y'(0) = \frac{-1}{\sin(1)}$

$$y'(t) = -\frac{\cos(t)}{\sin(1)}$$

so $y'(0) = -\frac{\cos(0)}{\sin(1)} = \frac{-1}{\sin(1)}$.

2.1.2 Using results from lecturenotes

Another way to solve analytically the O.D.E. $y'' + y = 1$ is to use the theorem (lecturenotes 7/11/2007) on O.D.E. with Linear and constant-coefficients. This theorem gives you the solution $y_1(t) = \cos(t)$ and $y_2(t) = \sin(t)$ to the homogeneous equation $y'' + y = 0$. You can also guess easily a particular solution to the original O.D.E. $y'' + y = 1$ by taking $y_p(t) = 1$.

So the general solution to $y'' + y = 1$ is:

$$y(t) = A\,y_1(t) + B\,y_2(t) + y_p(t) = A\ \cos(t) + B\ \sin(t) + 1$$

where A and B are constant that you can find using the boundary conditions:

$$y(0) = A\cos(0) + B\sin(0) + 1 = A + 1 = 1$$

so $A = 0$, and

$$y(1) = B\sin(1) + 1 = 0$$

so $B = \frac{-1}{\sin(1)}$. So the exact solution is:

$$y(t) = 1 - \frac{\sin(t)}{\sin(1)}$$

2.2 Numerical solutions

This will be done in class on 28-29 of November.