Assignment I - CS7ET01

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1 Part I: Linear Algebra, Least Squares and Principal Component Analysis

1. **Linear Algebra**[1]

   (a) Consider the matrix

   $$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$$

      i. Compute the determinant of A.
      ii. Why is A invertible?
      iii. Compute A^{-1}.
      iv. Verify your results in questions 1(a)i and 1(a)iii using Matlab.

      Hint: use commands det() and inv(). To get information or help about how to use a command, type 'help' <command>, e.g. 'help det'.

   (b) Consider the matrix

   $$C = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

      i. Find the eigenvalues and associated eigenvectors of C.
      ii. Verify your result with matlab using the command $svd()$.

   (c) Show that[2]:

   $$\frac{\partial(\mathbf{x}^T A\mathbf{x})}{\partial\mathbf{x}} = A\mathbf{x} + A^T\mathbf{x} \qquad (1)$$

   (d) Read sections 2.6 Singular Value Decomposition and 2.9 Cholesky Decomposition, in the Numerical recipes online at http://www.nrbook.com/a/bookcpdf.php. Answer the following questions (be concise, possibly used figures and diagrams):

      i. Explain Singular Value Decomposition (SVD).
      ii. Can SVD be applied to a square matrix?
      iii. What does the Cholesky Decomposition do?

[1] Questions 1(a) & 1(b) are here to refresh your mind from what you should remember from your BSc. A good reference is the book on linear algebra by Jim Hefferon available at http://joshua.smcvt.edu/linearalgebra/, in particular chapter I, section 3.1 and chapter 3 section 4.9.

[2] c.f. Tutorial 18/10/2007.

iv. What is the command in matlab doing the Cholesky Decomposition?

Hint: type 'lookfor cholesky' in matlab command window, and a list of related functions should appear.

v. What are the differences between SVD and the Cholesky Decomposition? (cite at least 2 differences)

vi. Propose a question and answer it on either the SVD or the Cholesky decomposition.

2. **Least square estimation.**

(a) The least square solution to the equation $\mathrm{A}\mathbf{x} = \mathbf{y}$ is:

$$\mathbf{x}_{LS} = (\mathrm{A}^T\mathrm{A})^{-1}\mathrm{A}^T\mathbf{y}$$

Show that we have the same result as before for $\mathbf{x}_{LS}$ using this formulation:

$$\mathbf{x}_{LS} = \arg\min_{\mathbf{x}} \left\{ \mathcal{J}(\mathbf{x}) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{j,i} x_i - y_j \right)^2 \right\}$$

(b) Read *Least-Squares Meshes*, by O. Sorkine and D. Cohen-Or, proceeding of shape Modeling International, pp. 191-199, 2004, available at http://cg.cs.tu-berlin.de/~sorkine/ProjectPages/LsMeshes/lsmesh.pdf. Answer the following questions:

i. What is this article presenting?

ii. Define what is a vertex (or vertices).

iii. Describe how least squares estimation is used in this paper.

iv. Describe the different steps you would do for programming the process described in this paper.

v. Is there any mathematics you dont understand in this paper?

(c) Imagine you observe a set of points $\{(x_i, y_i)\}$ of a variable $(x, y) \in \mathbb{R}^2$ that follows the equation:

$$ax^2 + by^2 + cxy + dx + ey - 1 = 0 \quad \text{(ellipse)}$$

i. Propose an algorithm to estimate the values (a, b, c, d, e).

ii. Propose a Matlab program to solve this problem[3].

3. **Principal component analysis.**

(a) Show that:

$$\begin{aligned} \|\mathbf{v}\mathbf{v}^T\mathbf{x}_i - \mathbf{v}\mathbf{v}^T\mathbf{x}_j\|^2 &= (\mathbf{v}\mathbf{v}^T\mathbf{x}_i - \mathbf{v}\mathbf{v}^T\mathbf{x}_j)^T(\mathbf{v}\mathbf{v}^T\mathbf{x}_i - \mathbf{v}\mathbf{v}^T\mathbf{x}_j) \\ &= \mathbf{v}^T(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T\mathbf{v} \end{aligned}$$

(b) Show that:

$$\begin{aligned} \mathcal{J}(\mathbf{v}) &= \tfrac{1}{2N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{v}^T(\mathbf{x}_i - \mathbf{x}_j)(\mathbf{x}_i - \mathbf{x}_j)^T\mathbf{v} \\ &= \mathbf{v}^T\mathrm{C}\mathbf{v} \end{aligned}$$

with C the covariance matrix of the set $\{\mathbf{x}_i\}_{i=1\cdots N}$.

(c) Using results in page D4 on the appendix D, find the solution $\mathbf{v}_{pca}$ such that:

$$\mathbf{v}_{pca} = \arg\max_{\mathbf{v}} \{\mathbf{v}^T\mathrm{C}\mathbf{v} - \lambda(\mathbf{v}^T\mathbf{v} - 1)\}$$

4. **Using Lagrange multipliers**

(a) Least squares with Constraint. Solve the following problem:

$$\begin{aligned} &\arg\min_{\mathbf{x}} \|\mathrm{A}\mathbf{x} - \mathbf{b}\|^2 \\ &\text{subject to } \mathrm{C}\mathbf{x} = \mathrm{D} \end{aligned}$$

where $\mathbf{x}$ and $\mathbf{b}$ vectors; A, C and D are matrices.

(b) Read *Practical Least Squares for Computer Graphics*, F. Pighin & J. P. Lewis, SIGGRAPH course notes 2007, section 6, available at http://graphics.stanford.edu/~jplewis/lscourse/ls.pdf

i. What are the applications of Lagrange multipliers that are cited in that section?

[3]you can take example of the program leastsquare.m given to you on the course webpage.

2 Part II: Ordinary Differential Equations

1. **Solve**

   (a) analytically $y'' + ty'^2 = 0$

   (b) analytically $y'' = -ky$.

   (c) numerically the equation $y'' = -ky$ on the interval $t \in [0; \sqrt{k} \times 2\pi]$.

2. Have a look at the following matlab tutorial on Linear Algebra and Music `http://web.mit.edu/18.06/www/Essays/linear-algebra-and-music.pdf`:

   (a) Tell me what is the equation $y'' = -ky$ representating in that example?

   (b) Do section 3 in this tutorial on Linear Algebra and Music.

3. **Laplace Transform.**

   (a) Find[4] the Laplace Transform of:

      i. $f(t) = 6\exp^{(-5t)} + \exp^{(3t)} + 5t^3 - 9$

      ii. $f(t) = 4\cos(4t) - 9\sin(4t)$

      iii. $f(t) = t^{3/2}$

      iv. $f(t) = t^2 \sin(2t)$

   (b) Find the inverse of the Laplace transform of

      i. $F(s) = \frac{10}{s^2+10s+16}$

   (c) Solve using Laplace transform:

      i. $y'' + 3ty' - 6y = 2$

[4] using direct integration or the table of Laplace transforms.