Ordinary Differential Equations I

Definition

An Ordinary differential equation (ODE) is an equation (or system of equations) written in term of an unknown function y and its derivatives with respect to a single independent variable t:

$$F(t, y, y', \cdots, y^{(n-1)}) = y^{(n)}$$

Example

- Newton's second law (m mass , f forces, y position):

$$m\frac{d^2y}{dt^2} = f(y)$$

- Electrical circuits

Ordinary Differential Equations II

Definition

The order of a differential equation is the highest derivative that occurs.

Example

An example of ODE of order 1:

$$\frac{dy}{dt} = f(t)$$

It is easily solved by integration:

$$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t)dt$$

Ordinary Differential Equations III

Definition

A Linear ODE of order N can be written as:

$$\frac{d^Ny}{dt^N} + A_1(t)\frac{d^{N-1}y}{dt^{N-1}} + \cdots + A_{N-1}(t)\frac{dy}{dt} + A_N(t)y = f(t)$$

Ordinary Differential Equations IV

Definition

An ODE is said homogeneous if only it is expressed w.r.t. the unknown function (e.g. y). If it is also depending of a function $f(t)$, then the ODE is said inhomogeneous.

If the unknown function y does not appear within powers or more complex functions , then the differential equation is said linear.

Example

$$\frac{dy}{dt} + y = f(t) \begin{cases} if\ f(t) = 0, \text{homogeneous \& linear ODE} \\ if\ f(t) \neq 0, \text{inhomogeneous \& linear ODE} \end{cases}$$

First Order Equations I

Definition

An ODE of the first order is an equation of the form:

$$F(t, y, y') = 0$$

We consider 3 types:

1. Separable equations
2. Linear equations
3. Bernouilli equations

First Order Equations II

Definition (Separable equations)

A first-order equation is separable if:

$$y' = f(t)g(y)$$

or

$$\frac{dy}{g(y)} = f(t)dt$$

Example (Separable equations: The Growth Decay equation)

Consider $y(t)$, the number of individuals in a population at time t. β is the birth rate and δ is the death rate.

$$y' = (\beta - \delta)y$$

$(\beta - \delta)$ is the growth rate.

First Order Equations III

Definition (First order linear equation)

A first order linear equation is of the form:

$$y' + p(t)y = q(t)$$

This can be solved by multiplying by:

$$\exp\left(\int_a^t p(s)ds\right)$$

and the equation becomes:

$$\frac{d\left(y\exp\left(\int_a^t p(s)ds\right)\right)}{dt} = q(t)\exp\left(\int_a^t p(s)ds\right)$$

Then both side can be integrated from a to t.

First Order Equations IV

Definition (Bernouilli equations)

A Bernouilli equation is a non linear equation:

$$y' + p(t)\ y = q(t)\ y^n$$

the transformation $w = y^{1-n}$ turns a Bernouilli equation into a linear equation in w.

First Order Equations V

Example (Bernouilli)

Find an expression for the solution to the initial value problem:

$$y' + 2ty = \sqrt{t} \quad , \ y(0) = 3$$

Solution: the integrating factor is $\exp(\int_0^t 2sds) = \exp(t^2)$ so the equation to solve becomes:

$$(y\exp(t^2))' = \sqrt{t}\exp(t^2)$$

or

$$y(t)\exp(t^2) - y(0) = \int_0^t \sqrt{s}\exp(s^2)ds$$

Exercises

Solve:

1. $y' + 2y = \exp(-t)$
2. $y' = -3y$
3. $y' - ty = t^2y^2$