3 Example: Galerkin Formulation of Poisson's Equation

We consider the Poisson equation:

$$u''(x) = -f(x) \quad 0 < x < 1$$

with boundary conditions $u(0) = u(1) = 0$. We want to estimate the trial solution

$$\tilde{u}(x) = \sum_{i=1}^{N} \alpha_i \Phi_i(x)$$

using the Galerkin method and using the piecewise linear functions (cf. figure 1) as basis:

$$\Phi_j(x) = \begin{cases} \frac{1}{\Delta}(x - x_{j-1}) & \text{for } x_{j-1} \leq x \leq x_j \\ \frac{1}{\Delta}(x_{j+1} - x) & \text{for } x_j \leq x \leq x_{j+1} \\ 0 \quad \text{otherwise} \end{cases} \tag{3}$$

with $\Delta = x_{j+1} - x_j, \quad \forall j$, with $0 = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = 1$.

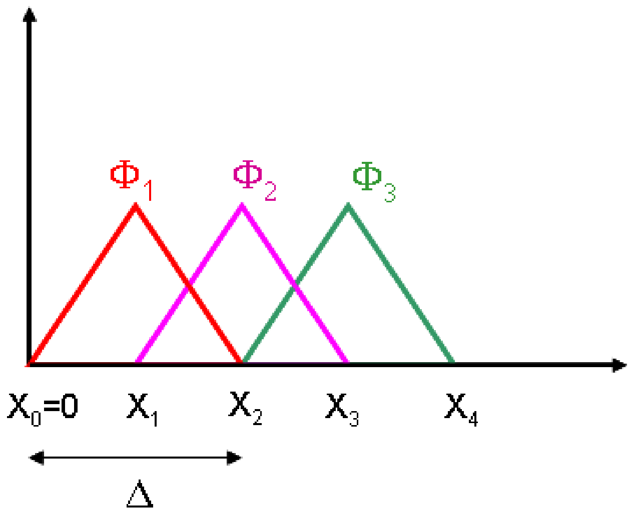

Figure 1: Linear basis functions.

One can notice that

$$\Phi_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

or $\Phi_j(x_i) = \delta_{ij}$ (Kronecker's delta).

Considering the boundary conditions, we do have $\Phi_j(0) = \Phi_j(1) = 0, \quad \forall j = 1 \cdots N$, hence the boundary conditions are verified for the trial solution $\tilde{u}(0) = \tilde{u}(1) = 0$.

3.1 Residual function

The residual function is:

$$\begin{aligned} R(x;\alpha) &= \tilde{u}''(x) + f(x) \\ &= \textstyle\sum_{i=1}^{N} \alpha_i \Phi_i''(x) + f(x) \end{aligned} \tag{4}$$

3.2 Solving with Galerkin method

We need to find α such that:

$$\int_0^1 R(x;\alpha)\, \Phi_j(x)\, dx = 0 \quad \forall j$$

Replacing the residual by its expression (4), we have

$$\int_0^1 \left(\sum_{i=1}^{N} \alpha_i \Phi_i^{''}(x) + f(x) \right) \; \Phi_j(x)\; dx = 0 \quad \forall j$$

or

$$\sum_{i=1}^{N} \alpha_i \int_0^1 \Phi_i^{''}(x)\; \Phi_j(x)\; dx = - \int_0^1 f(x)\; \Phi_j(x)\; dx \quad \forall j \tag{5}$$

3.2.1 Consider the integral $\int_0^1 \Phi_i^{''}(x)\; \Phi_j(x)\; dx$

We can solve it by integration by parts:

$$\int_0^1 \Phi_i^{''}(x)\; \Phi_j(x)\; dx = \left[\Phi_i^{'}(x)\; \Phi_j(x) \right]_0^1 - \int_0^1 \Phi_i^{'}(x)\; \Phi_j^{'}(x)\; dx$$

We have the first term $\left[\Phi_i^{'}(x)\; \Phi_j(x) \right]_0^1 = 0$ since $\Phi_j(0) = \Phi_j(1) = 0, \quad \forall j$. From equation (3), we can compute the derivative of Φ_j:

$$\Phi_j^{'}(x) = \begin{cases} \frac{1}{\Delta} & \text{for} \quad x_{j-1} \leq x \leq x_j \\ \frac{-1}{\Delta} & \text{for} \quad x_j \leq x \leq x_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

We can note that $\Phi_j^{'}$ is discontinuous at x_j, x_{j-1} and x_{j+1}. We compute the term $\int_0^1 \Phi_i^{'}(x)\; \Phi_j^{'}(x)\; dx$ in all cases:

- if $i = j$ then
$$\int_0^1 \Phi_j^{'}(x)\; \Phi_j^{'}(x)\; dx = \frac{2}{\Delta}$$
- if $i = j - 1$ or $i = j + 1$
$$\int_0^1 \Phi_{j-1}^{'}(x)\; \Phi_j^{'}(x)\; dx = \frac{-1}{\Delta}$$
- otherwise
$$\int_0^1 \Phi_i^{'}(x)\; \Phi_j^{'}(x)\; dx = 0$$

3.2.2 Solving equation (5)

Using linear algebra, we store all the values

$$< f, \Phi_j > \quad = \quad \int_0^1 f(x)\; \Phi_j(x)\; dx$$

into a a vector $\mathbf{b}$ (i.e. such that the j component of $\mathbf{b}$ is$b_j =< f, \Phi_j >$). All the values $< \Phi_i^{'}, \Phi_j^{'} >= \int_0^1 \Phi_i^{'}(x)\; \Phi_j^{'}(x)\; dx$ are stored in a $N \times N$ matrix A such that its components are defined by $a_{ij} =< \Phi_i^{'}, \Phi_j^{'} >$.

Then we can rewrite equation (5):

$$\mathrm{A}\boldsymbol{\alpha} = \mathbf{b}$$

or

$$\frac{1}{\Delta} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & 2 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & -1 & 2 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_N \end{pmatrix}$$

In this linear equation, the unknown we are looking for is α and it can easily be computed:

$$\hat{\boldsymbol{\alpha}} = \mathrm{A}^{-1}\mathbf{b}$$