Introduction

Today[2]: Kalman Filters

Definition (time series)

Time series is a sequence of data points index by time:

$$\{X_1, X_2, \cdots\} = \{X_t\}_{t=1\cdots T}$$

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Kalman Filter I

Given a sequence of observations: $\{y_1, y_2, \cdots, y_T\}$ and the following p.d.fs.:

$$\begin{cases} f(x_1) = \mathcal{N}(\mu_1, \sigma_1^2) \\ f(x_t|x_{t-1}) = \mathcal{N}(\lambda x_{t-1}, \sigma^2) \\ f(y_t|x_t) = \mathcal{N}(x_t, \sigma_y^2) \end{cases}$$

Estimate the posterior $f_{x_t|y_{1:t}}(x_t|y_{1:t})$ **?**

Kalman Filter II

Step 1: express $f(x_t|y_{1:t})$ w.r.t. $f(x_{t-1}|y_{1:t-1})$

Kalman Filter III

$$f(x_t|y_{1:t}) = \frac{f(y_{1:t-1})}{f(y_{1:t})}\ f(y_t|x_t) \int f(x_t|x_{t-1})\ f(x_{t-1}|y_{1:t-1})\ dx_{t-1}$$

Kalman Filter IV

Step 2: Compute the integral using the hypotheses,

Kalman Filter V

$$\begin{cases} \mu_t = \lambda\ (1 - K_t)\ \mu_{t-1} + K_t\ y_t \\ \sigma_t^2 = \sigma_y^2\ K_t \\ K_t = \frac{\sigma^2 + \lambda^2 \sigma_{t-1}^2}{\sigma^2 + \sigma_y^2 + \lambda^2 \sigma_{t-1}^2} \end{cases}$$

Remarks:

In the general case we have the following integral to solve:

$$f(x_t|y_{1:t}) = \frac{f(y_{1:t-1})}{f(y_{1:t})}\ f(y_t|x_t) \int f(x_t|x_{t-1})\ f(x_{t-1}|y_{1:t-1})\ dx_{t-1}$$

1. Prediction:
$$f(x_t|y_{1:t-1}) = \int f(x_t|x_{t-1})\ f(x_{t-1}|y_{1:t-1})\ dx_{t-1}$$
2. Update
$$f(x_t|y_{1:t}) \propto f(y_t|x_t)\ f(x_t|y_{1:t-1})$$