Finite Element Method I

Example

Lets consider the O.D.E. on the interval $t \in \Delta = [0;1]$

$$f(t, y, y') = y'(t) + y(t) = 0$$

with the initial condition $y(0) = 1$.

We can solve this O.D.E.:

- analytically giving the exact solution:

$$y(t) = \exp(-t)$$

- numerically giving an approximate solution using:
  - Finite Difference Method,
  - Finite Element Method.

Finite Element Method II

Definition (Finite Element Method)

Consider an equation (O.D.E.) $f(t, y, y') = 0$ of a function $y(t)$ defined on the interval $t \in \Delta$:

1. Define a trial solution of the form:

$$\tilde{y}(t) = \phi_0 + \sum_{i=1}^{n} a_i \phi_i(t)$$

   where the functions $\{\phi_i(t)\}$ are basis functions, and $\phi_0$refers to a constant that usually can be inferred from initial conditions.

2. Replace $y(t)$ by $\tilde{y}(t)$ in the O.D.E. $f(t, \tilde{y}(t), \tilde{y}'(t)) = R(t, \mathbf{a})$ (R is called the residual) and solve the coefficients $\{a_i\}_{i=1\cdots n}$.

The estimated $\tilde{y}(t)$ gives a numerical solution to the O.D.E. defined on the interval Δ.

Solving $y' + y = 0$ using collocation method I

Definition (Collocation method)

The collocation method imposes that the residual vanishes at n points i.e. $R(t_i; \mathbf{a}) = 0$ for $t_i \in \Delta$ and $i = 1, \cdots, n$.

Example

We choose the basis $\phi_i(t) = t^i$ and $n = 2$ so:

$$\tilde{y}(t) = 1 + a_1\ \phi_1(t) + a_2\ \phi_2(t) = 1 + a_1\ t + a_2\ t^2$$

and

$$R(t, \mathbf{a}) = 1 + (1 + t)\ a_1 + (2t + t^2)\ a_2$$

Solving $y' + y = 0$ using collocation method II

Example

Lets take $t_1 = \frac{1}{3}$ and $t_2 = \frac{2}{3}$ such that we impose $R(t_1, \mathbf{a}) = R(t_2, \mathbf{a}) = 0$. This gives us 2 equations:

$$\begin{cases} 1 + \left(1 + \frac{1}{3}\right)\ a_1 + \left(\frac{2}{3} + \left(\frac{1}{3}\right)^2\right)\ a_2 = 0 \\ 1 + \left(1 + \frac{2}{3}\right)\ a_1 + \left(\frac{4}{3} + \left(\frac{2}{3}\right)^2\right)\ a_2 = 0 \end{cases}$$

Solving in a_1 and a_2 gives:

$$\tilde{y}(t) = 1 - \frac{27}{29} t + \frac{9}{29} t^2$$

Compare that with the true function $y(t) = \exp(-t)$.

Sub-domain method I

Definition (subdomain method)

Divide the interval Δ of definition of the O.D.E. into subdomains $\{\Delta_i\}_{i=1,\cdots,n}$, and solve $\mathbf{a}$ such that :

$$\frac{1}{\Delta_i} \int_{\Delta_i} R(t; \mathbf{a})\ dt = 0 \quad \forall i = 0, \cdots, n$$

Sub-domain method II

Example

We choose $\Delta_1 = [0; \frac{1}{2}]$ and $\Delta_2 = [\frac{1}{2}; 1]$ and :

$$\frac{1}{\Delta_1} \int_{\Delta_1} R(t; \mathbf{a})\ dt = 2 \left[\frac{1}{2} + \frac{5}{8}\ a_1 + \frac{7}{24}\ a_2\right] = 0$$

and

$$\frac{1}{\Delta_2} \int_{\Delta_2} R(t; \mathbf{a})\ dt = 2 \left[\frac{1}{2} + \frac{7}{8}\ a_1 + \frac{25}{24}\ a_2\right] = 0$$

Solving this system gives the solution

$$\tilde{y}(t) = 1 - \frac{18}{19}\ t + \frac{6}{19}\ t^2$$

Compare $\tilde{y}(t)$ estimated using subdomain method, with the exact solution $y(t)$.

Least-Squares Method I

Definition (Least-Squares Method)

We can use least square estimation to find the optimum values of the parameters $\mathbf{a}$:

$$\hat{\mathbf{a}} = \arg\min \left\{ \int_{\Delta} (R(t;\mathbf{a}))^2 \, dt \right\}$$

or solve

$$\frac{\partial}{\partial a_i} \int_{\Delta} (R(t;\mathbf{a}))^2 \, dt = 0 \quad \forall i$$

or

$$\int_{\Delta} \frac{\partial}{\partial a_i} (R(t;\mathbf{a}))^2 \, dt = 0 \quad \forall i$$

or

$$\int_{\Delta} 2 \frac{\partial R(t;\mathbf{a})}{\partial a_i} \, R(t;\mathbf{a}) \, dt = 0 \quad \forall i$$

Least-Squares Method II

Example

Exercise: Try to find $\tilde{y}$ with the Least squares method.

Galerkin Method I

Definition (Galerkin Method)

The Galerkin method imposes the following criterium to estimate $\mathbf{a}$:

$$\int_{\Delta} R(t;\mathbf{a}) \, \phi_i(t) dt = 0 \quad \forall i = 1, \cdots, n$$

Example

Find that the estimate of $\tilde{y}(t)$ using Galerkin method is:

$$\tilde{y}(t) = 1 - \frac{32}{35} \, t + \frac{2}{7} \, t^2$$

Compare that solution with the exact one $y(t)$.

Exercises

Solve on the interval $[0;1]$ the O.D.E.:

$$y''(t) + y(t) = 1$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$:

1. analytically.
2. numerically using
   1. the collocation method
   2. the subdomain method
   3. the Galerkin method
3. Draw all the solutions you found on a graph (use matlab).

Method of Weighted residuals I

Definition (Weighted residuals)

In fact all the previous method (collocation, galerkin, subdomain, Least squares) can be understood as special cases of the Method of Weighted residuals (MWR). The notion of MWR is to force the residual $R(t;\mathbf{a})$ to zero in some average sense over the domain. That is:

$$\int_{\Delta} R(t,\mathbf{a}) \, W_i(t) \, dt = 0 \qquad i = 1, \cdots, n$$

where the number of weight functions $W_i(t)$ is exactly equal to the number of unknown constant a_i in $\tilde{y}$.

Method of Weighted residuals II

Weighting function W_i for:

1. the Collocation method:
$$W_i(t) = \delta(t - t_i)$$
2. the Sub-domain method:
$$W_i(t) = \begin{cases} 1 & \text{if } t \in \Delta_i \\ 0 & \text{otherwise} \end{cases}$$
3. the Least Squares method:
$$W_i(t) = 2 \frac{\partial R(t;\mathbf{a})}{\partial a_i}$$
4. the Galerkin Method:
$$W_i(t) = \frac{\partial \tilde{y}}{\partial a_i} = \phi_i(t)$$

Remarks on FDM and FEM

Finite Difference Method: FDM approximates an operator (e.g., the derivative) and solves a problem on a set of points (the grid).

Finite Element Method: FEM uses exact operators but approximates the solution basis functions. Also, FE solves a problem on the interiors of grid cells (and optionally on the gridpoints as well).

Which basis functions? I

- Amongst the methods of weighted residuals, the Galerkin method is the most commonly used.
- MWR apply equally well to any family of basis functions $\{\phi_i\}$. However to be sucessful, the basis functions need to be chosen with care
  - Global basis functions perform badly in fitting local behaviour, e.g.:

$$\phi_i(t) = t^i$$

Which basis functions? II

- More local basis functions can be defined so that there are non-zero only on small regions, e.g. piecewise linear (or hat) functions:

$$\phi_i(t) = \begin{cases} \frac{t-t_{i-1}}{t_i-t_{i-1}} & t_{i-1} \le t < t_i \\ \frac{t_{i+1}-t}{t_{i+1}-t_i} & t_i \le t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Which basis functions? III