Numerical solution of equation

- Linear equations can be easily solved by basic algebra.

Example: Find the solution(s) of $f(x) = 0$ with the linear function $f(x) = \sqrt{2} \cdot x - 3$.
Set of solutions: $\left\{ \frac{3}{\sqrt{2}} \right\}$

- Non-linear equation such as quadratic equations that can be solved

Example: Find the solution(s) of $f(x) = 0$ with $f(x) = a\,x^2 + b\,x + c$.
Set of solutions: $\left\{ \frac{-b+\sqrt{b^2-4ac}}{2a}, \frac{-b-\sqrt{b^2-4ac}}{2a} \right\}$

- Lets consider the non-linear equation $f(x) = 0$. How do we compute the root r of this equation ?

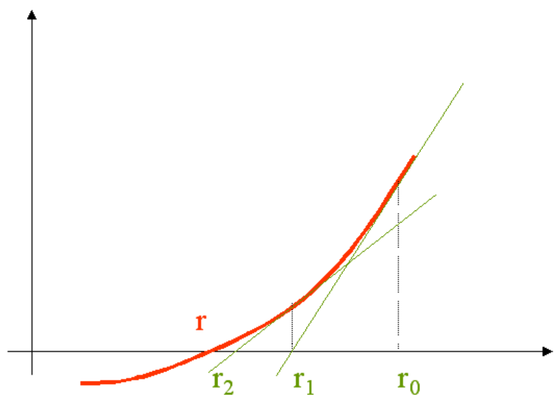
Newton-Raphson method

Newton-Raphson method.
Let r be the root of a non-linear equation $f(x) = 0$. Starting from an initial guess r_0, the sequence defined as:

$$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)}$$

is converging toward r. Using a computer, you use a *for* loop until the iteration n such as r_n is close enough to r (i.e. depending of the accuracy required).

Newton-Raphson method



Newton-Raphson method

To start the Newton-Raphson procedure, you need to choose an appropriate starting value r_0 not far from the solution r. You can :

1. plot a graph of the function and see approximately where the roots lie,
2. or evaluate the function at some obvious values.

The Newton-Raphson method normally requires a close initial estimate to the actual root otherwise it may fail.

Example: Newton-Raphson method

We want to compute (with a computer) $\sqrt{a}$ for a known number $a > 0$.

1. Using the Newton-Raphson procedure, imagine a program using only the basic operations $+ - / \times$ to compute $\sqrt{a}$.
2. Compute then $\sqrt{2}$ (i.e. $a = 2$) with 2 decimal precision starting from $r_0 = 3$.
3. Prove by induction that the sequence is $\{r_n\}$ defined in question 2, is bounded from below by $\sqrt{a}$.
4. Prove that the sequence $\{r_n\}$ defined in question 2, is monotonically decreasing.

Example: Newton-Raphson method I

1. Lets define $f(x) = x^2 - a$. The root r of $f(x) = 0$ is $r = \sqrt{a}$.
   Using the Newton-Raphson method we know that:
   $$r_{n+1} = r_n - \frac{f(r_n)}{f'(r_n)} \quad \Rightarrow \quad r_{n+1} = r_n - \frac{r_n^2 - a}{2r_n}$$
   converge to the root $r = \sqrt{a}$ such $f(r) = 0$. Example of program:
   - RzDNewtonRaphson.m
2. Starting with $r_0 = 3$. We get $r_1 = 1.83$, $r_2 = 1.46$, $r_3 = 1.42$, $r_4 = 1.41$ and $r_5 = 1.41$.
   We can then approximate $\sqrt{2} = 1.41$.

Example: Newton-Raphson method II

3. We know $r_0 = 3 > \sqrt{a}$ (when $a = 2$) and assuming $r_n > \sqrt{a}$, then we compute:

$$
\begin{aligned}
r_{n+1} - \sqrt{a} &= r_n - \tfrac{r_n^2 - a}{2r_n} - \sqrt{a} \\
&= \tfrac{2r_n^2 - r_n^2 - a - \sqrt{a}2r_n}{2r_n} \\
&= \tfrac{r_n^2 - 2\sqrt{a}r_n - a}{2r_n} \\
&= \tfrac{(r_n - \sqrt{a})^2}{2r_n} \geqslant 0
\end{aligned}
$$

Then $r_{n+1} > \sqrt{a}$. Hence by induction, we have $\forall n \; r_n > \sqrt{a}$ (the sequence is bounded from below).

Example: Newton-Raphson method III

4. We compute:

$$
\begin{aligned}
r_{n+1} - r_n &= r_n - \tfrac{r_n^2 - a}{2r_n} - r_n \\
&= -\tfrac{r_n^2 - a}{2r_n}
\end{aligned}
$$

From the previous question, we know that $\forall n, \quad r_n \geqslant \sqrt{a}$. Then $r_n^2 - a \geqslant 0$, and $\forall n, \quad r_{n+1} - r_n \leqslant 0$ (The sequence is decreasing)

Exercises: Newton-Raphson method

1. Apply the Newton-Raphson procedure with the first approximation $r_0 = -2$, to find a root of the following equations to two decimal places:
   1. $x^3 - 2x + 7 = 0$
   2. $x^4 - 3x^2 - 2$
   3. $e^x - 2x - 5 = 0$

2. The horizontal distance x travelled by a projectile is given by:

$$x = \frac{u^2}{25} \sin(\theta) \cdot \cos(\theta), 0 < \theta < \frac{\pi}{2}$$

where u is the initial velocity and θ is the angle that the projectile makes with the horizontal. Determine the value of θ which maximizes x.