Using the Fourier series expansion I

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x < \pi \end{cases}$$

You can compute its Fourier expansion:

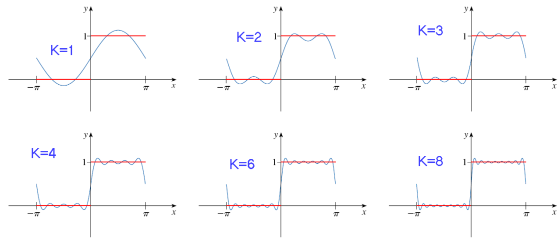
$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin((2k-1)x)$$

Now consider the estimate:

$$\hat{f}_K(x) = \frac{1}{2} + \sum_{k=1}^{K} \frac{2}{(2k-1)\pi} \sin((2k-1)x)$$

Role of K ?

Using the Fourier series expansion II



$\hat{f}_K$ for different K.

Using the Fourier series expansion III

Theorem (Fourier Convergence Theorem)

If f is a periodic function with period $2L$ and f and f' are piecewise continuous on $[-L;L]$, then its Fourier series is convergent. The sum of the Fourier series is equal to $f(x)$ at all numbers where f is continuous. At the numbers where is discontinuous, the sum of the Fourier series is the average of the right and left limits, that is:

$$\frac{1}{2}[f(x^+) + f(x^-)]$$

Exercise. Find the Fourier series of the triangular wave function defined by $f(x) = |x|$ for $-1 \le x \le 1$ and $f(x+2) = f(x)$ for all x. For which values of is equal to the sum of its Fourier series?

Transient signals

Our attention is more attracted by transients and movements as opposed to stationary events.

- Fourier analysis has focused on stationary signals.
- Wavelets are better designed for the analysis of transient signals.

Uncertainty Principle I

Theorem

The uncertainty principle states that the energy spread of a function and its Fourier transform cannot be simultaneously arbitrarily small.

Example

Example of a signal f well localised in time:

$$f(t) = \delta(t - t_0) \quad \leftrightarrow \quad F(\omega) = \exp(-\imath\omega t_0)$$

but the energy $\|F(\omega)\|^2 = 1$ is uniform in the ω-space.

Windowed Fourier I

Definition

Gabor atoms are constructed by translating in time and frequency a time window g:

$$g_{u,\xi}(t) = g_{u,\xi}(t-u)\ \exp(\imath\xi t)$$

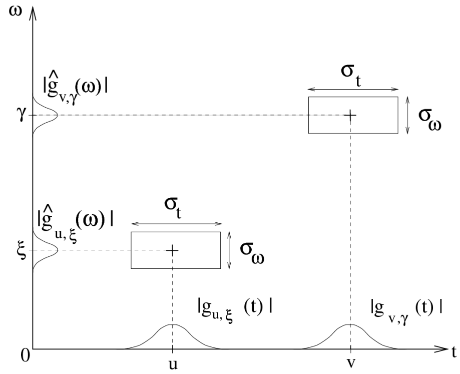
The energy of $g_{u,\xi}$ is concentrated in the neighborhood of u over an interval size σ_t. Its Fourier transform is a translation by ξ of the Fourier transform G of g:

$$G_{u,\xi}(\omega) = G(\omega - \xi)\exp(-iu(\omega - \xi))$$

The energy of $G_{u,\xi}$ is therefore localised near the frequency ξ over an interval size σ_ω. The Heisenberg rectangle is centered on (u,ξ) with sides $(\sigma_t, \sigma_\omega)$. The uncertainty principle states that its area satisfies:

$$\sigma_t \sigma_\omega \ge \frac{1}{2}$$

Windowed Fourier II



Time Frequency boxes (from *a wavelet tour of signal processing*, S. Mallat.) .

Windowed Fourier III

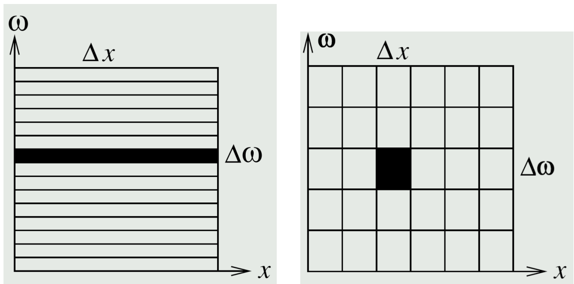
The area $\sigma_t \sigma_\omega$ is minimum when g is a Gaussian in which case $g_{u,\xi}$ are called the Gabor function.

Definition

The windowed Fourier transform defined by Gabor correlates a signal f:

$$Sf(u,\xi) = \int_{-\infty}^{+\infty} f(t)\; \overline{g}_{u,\xi}(t) dt = \int_{-\infty}^{+\infty} f(t)\; \overline{g}_{u,\xi}(t-u)\; \exp(-\imath \xi t)\; dt$$

Windowed Fourier IV



FT WFT

Introduction to wavelets I

Definition (Wave)

A wave is defined as an oscillating function of time or space, such as a sinusoid.

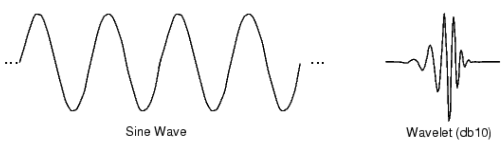
Example (Fourier)

Fourier analysis is wave analysis. It is useful to study periodic, time-invariant, or stationnary phenomena.

Definition

A wavelet is a small wave which has its energy concentrated in time. It is a a tool to analyse transient, nonstationnary or time varying phenomena.

Introduction to wavelets II



Wavelets and wavelet expansion system I

A function $f(t)$ is often better analysed and processed when expressed as a linear combination:

$$f(t) = \sum_l a_l \psi_l(t)$$

with l an integer index for the finite or infinite sum, $\{a_l\}$ are the real value expansion coefficients, and $\{\psi_l(t)\}$ are a basis of functions. For orthogonal basis functions:

$$< \psi_k(t) | \psi_l(t) >= \int \overline{\psi}_k(t) \psi_l(t) dt = 0 \quad k \neq l$$

then the coefficients can be computed by the inner product:

$$a_k =< f(t) | \psi_k(t) >= \int \overline{f}(t) \psi_k(t) dt$$

Wavelets and wavelet expansion system II

Example (Example of expansion)

- For Fourier series, $\sin(k\omega_0 t)$ and $\cos(k\omega_0 t)$ are orthogonal basis.
- For Taylor series, t^k are the basis functions.

Wavelets and wavelet expansion system III

Definition (Discrete Wavelet expansion)

For a wavelet expansion, a 2 parameter system is constructed:

$$f(t) = \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} a_{j,k} \psi_{j,k}(t)$$

where j and k are integer indices and $\psi_{j,k}(t)$ are the wavelet expansion functions that usually form an orthogonal basis. The set of expansion coefficients $\{a_{j,k}\}$ are called the discrete wavelet transform of $f(t)$ and $a_{j,k}$ is computed by:

$$a_{j,k} = < f(t) | \psi_{j,k}(t) >$$

Wavelets and wavelet expansion system IV

Definition (Mother wavelet)

The first generation wavelet systems are generated from a single function or mother wavelet $\psi(t)$, by scaling and translation:

$$\psi_{j,k}(t) = \frac{1}{2^{j/2}} \psi\left(\frac{t - 2^j k}{2^j}\right) \quad (j,k) \in \mathbb{Z}^2$$

The space location is parameterized by k and j controls the scale.

Wavelets and wavelet expansion system V

Definition (Continuous Wavelet expansion)

The continuous wavelet transform of f at scale s and position u is defined as:

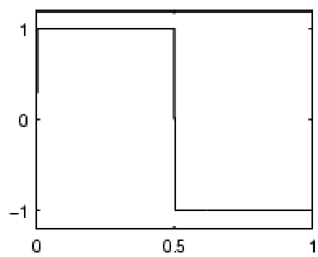
$$Wf(u,s) = \int_{-\infty}^{+\infty} f(t)\, \overline{\psi}_{u,s}(t)\, dt$$

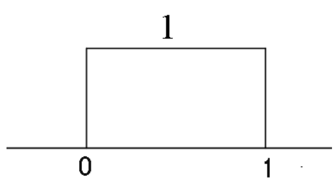
Like a windowed Fourier transform, a wavelet tranform can measure the time-frequency variations of spectral components, but it has a different time-frequency resolution.

Wavelets and wavelet expansion system VI

- In time, $\psi_{u,s}$ is centered on u with a spread proportional to s.
- The zooming capability of the wavelet transform allows to locate isolated singular events.
- The correspondence in between discrete and continuous wavelet transform are:
  - $2^j \leftrightarrow s$
  - $2^j k \leftrightarrow u$
- When discrete wavelets are used to transform a continuous signal the result will be a series of wavelet coefficients, and it is referred to as the wavelet series decomposition.

Haar wavelets I





Haar mother wavelet $\psi(t)$, and scaling function $\phi(t)$.

Haar wavelets II

The mother wavelet:

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1/2 \\ -1 & \text{if } 1/2 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and the scaling function

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Exercise:

1. Draw the basis function $\psi_{j,k}$ for different value of j and k.
2. Show

$$< \psi_{j,k} | \psi_{j',k'} > = \begin{cases} 0 & \text{if } (j,k) \neq (j',k') \\ 1 & \text{if } (j,k) = (j',k') \end{cases}$$

References

Wavelets for computer graphics: A primer (Part I & II) , E. J. Stollnitz, T. D. DeRose, and D. H. Salesin. , IEEE Computer Graphics and Applications, 1995.